Chapter 16
Superposition and Standing Waves

Conceptual Problems

1 • [SSM] Two rectangular wave pulses are traveling in opposite directions along a string. At $t = 0$, the two pulses are as shown in Figure 16-29. Sketch the wave functions for $t = 1.0$, $2.0$, and $3.0$ s.

**Picture the Problem** We can use the speeds of the pulses to determine their positions at the given times.

![Wave functions for $t = 0$, $1$, $2$, and $3$ s](image)

2 • Repeat Problem 1 for the case in which the pulse on the right is inverted.

**Picture the Problem** We can use the speeds of the pulses to determine their positions at the given times.

![Wave functions for $t = 0$, $1$, $2$, and $3$ s](image)

3 • Beats are produced by the superposition of two harmonic waves if
(a) their amplitudes and frequencies are equal, (b) their amplitudes are the same but their frequencies differ slightly, (c) their frequencies differ slightly even if their amplitudes are not equal, (d) their frequencies are equal but their amplitudes differ slightly.
Determine the Concept Beats are a consequence of the alternating constructive and destructive interference of waves due to slightly different frequencies. The amplitudes of the waves play no role in producing the beats. \( b \) and \( c \) are correct.

4 • Two tuning forks are struck and the sounds from each reach your ears at the same time. One sound has a frequency of 256 Hz, and the second sound has a frequency of 258 Hz. The underlying “hum” frequency that you hear is
(a) 2.0 Hz, (b) 256 Hz, (c) 258 Hz, (d) 257 Hz.

Determine the Concept The tone you hear is the average of the frequencies emitted by the vibrating tuning forks;
\[
\frac{1}{2}(f_1 + f_2) = \frac{1}{2}(256 \text{ Hz} + 258 \text{ Hz}) = 257 \text{ Hz}.
\]
Hence \( d \) is correct.

5 • In Problem 4, the beat frequency is (a) 2.0 Hz, (b) 256 Hz, (c) 258 Hz, (d) 257 Hz.

Determine the Concept The beat frequency is the difference between the two frequencies;
\[
\Delta f = f_1 - f_2 = 258 \text{ Hz} - 256 \text{ Hz} = 2 \text{ Hz}.
\]
Hence \( a \) is correct.

6 • As a graduate student, you are teaching your first physics lecture while the professor is away. To demonstrate interference of sound waves, you have set up two speakers that are driven coherently and in phase by the same frequency generator on the front desk. Each speaker generates sound with a 2.4-m wavelength. One student in the front row says she hears a very low volume (loudness) of the sound from the speakers compared to the volume of the sound she hears when only one speaker is generating sound. What could be the difference in the distance between her and each of the two speakers? (a) 1.2 m, (b) 2.4 m, (c) 4.8 m, (d) You cannot determine the difference in distances from the data given.

Determine the Concept Because the sound reaching her from the two speakers is very low, the sound waves must be interfering destructively (or nearly destructively) and the difference in distance between her position and the two speakers must be an odd multiple of a half wavelength. That is, it must be 1.2 m, 3.6 m, 6.0 m, etc. Hence \( a \) is correct.

7 • In Problem 6, determine the longest wavelength for which a student would hear “extra loud” sound due to constructive interference, assuming this student is located so that one speaker is 3.0 m further from her than the other speaker.
Determine the Concept Because the sound reaching the student from the two speakers is extra loud, the sound waves must be interfering constructively (or nearly constructively) and the difference in distance between the student’s position and the two speakers must be an integer multiple of a wavelength. Hence the wavelength of the sound is 3.0 m.

8 • Consider standing waves in an organ pipe. True or False:

(a) In a pipe open at both ends, the frequency of the third harmonic is three times that of the first harmonic.
(b) In a pipe open at both ends, the frequency of the fifth harmonic is five times that of the fundamental.
(c) In a pipe that is open at one end and stopped at the other, the even harmonics are not excited.

Explain your choices.

(a) True. If ℓ is the length of the pipe and v the speed of sound, the excited harmonics are given by \( f_n = \frac{n \nu}{2\ell} \), where \( n = 1, 2, 3 \ldots \) Hence the frequency of the third harmonic is three times that of the first harmonic.

(b) True. If ℓ is the length of the pipe and v the speed of sound, the excited harmonics are given by \( f_n = \frac{n \nu}{2\ell} \), where \( n = 1, 2, 3 \ldots \) Hence the frequency of the fifth harmonic is five times that of the first harmonic.

(c) True. If ℓ is the length of the pipe and v the speed of sound, the excited harmonics are given by \( f_n = \frac{n \nu}{4\ell} \), where \( n = 1, 3, 5 \ldots \)

9 • Standing waves result from the superposition of two waves that have (a) the same amplitude, frequency, and direction of propagation, (b) the same amplitude and frequency and opposite directions of propagation, (c) the same amplitude, slightly different frequencies, and the same direction of propagation, (d) the same amplitude, slightly different frequencies, and opposite directions of propagation.

Determine the Concept Standing waves are the consequence of the constructive interference of waves that have the same amplitude and frequency but are traveling in opposite directions. (b) is correct.
10 • If you blow air over the top of a fairly large drinking straw you can hear a fundamental frequency due to a standing wave being set up in the straw. What happens to the fundamental frequency, (a) if while blowing, you cover the bottom of the straw with your fingertip? (b) if while blowing you cut the straw in half with a pair of scissors? (c) Explain your answers to Parts (a) and (b).

**Determine the Concept**

(a) The fundamental frequency decreases.

(b) The fundamental frequency increases.

(c) **Part (a):** The fundamental frequency in a closed-open pipe is half that of an open-open pipe, so the frequency you hear with the bottom covered is half that you hear before you cover the bottom. **Part (b):** The fundamental frequencies of all pipes, independently of whether they are open-open or open-closed, varies inversely with the length of the pipe. Hence halving the length of the pipe doubles the fundamental frequency.

11 • [SSM] An organ pipe that is open at both ends has a fundamental frequency of 400 Hz. If one end of this pipe is now stopped, the fundamental frequency is (a) 200 Hz, (b) 400 Hz, (c) 546 Hz, (d) 800 Hz.

**Picture the Problem** The first harmonic displacement-wave pattern in an organ pipe open at both ends (open-open) and vibrating in its fundamental mode is represented in part (a) of the diagram. Part (b) of the diagram shows the wave pattern corresponding to the fundamental frequency for a pipe of the same length \( L \) that is stopped. Letting unprimed quantities refer to the open pipe and primed quantities refer to the stopped pipe, we can relate the wavelength and, hence, the frequency of the fundamental modes using \( v = f\lambda \).

Express the frequency of the first harmonic in the open-open pipe in terms of the speed and wavelength of the waves:

\[
f_1 = \frac{v}{\lambda_1}
\]
Relate the length of the open pipe to the wavelength of the fundamental mode:

\[ \lambda_1 = 2L \]

Substitute for \( \lambda_1 \) to obtain:

\[ f_1 = \frac{v}{2L} \]

Express the frequency of the first harmonic in the closed pipe in terms of the speed and wavelength of the waves:

\[ f_{1}' = \frac{v}{\lambda_1'} \]

Relate the length of the open-closed pipe to the wavelength of it's fundamental mode:

\[ \lambda_1' = 4L \]

Substitute for \( \lambda_1' \) to obtain:

\[ f_{1}' = \frac{v}{4L} = \frac{1}{2} \left( \frac{v}{2L} \right) = \frac{1}{2} f_1 \]

Substitute numerical values and evaluate \( f_{1}' \):

\[ f_{1}' = \frac{1}{2} (400 \text{ Hz}) = 200 \text{ Hz} \]

and \( (a) \) is correct.

12  ••  A string fixed at both ends resonates at a fundamental frequency of 180 Hz. Which of the following will reduce the fundamental frequency to 90 Hz? (a) Double the tension and double the length. (b) Halve the tension and keep the length and the mass per unit length fixed. (c) Keep the tension and the mass per unit length fixed and double the length. (d) Keep the tension and the mass per unit length fixed and halve the length.

**Picture the Problem** The frequency of the fundamental mode of vibration is directly proportional to the speed of waves on the string and inversely proportional to the wavelength which, in turn, is directly proportional to the length of the string. By expressing the fundamental frequency in terms of the length \( L \) of the string and the tension \( F \) in it we can examine the various changes in lengths and tension to determine which would halve it.

Express the dependence of the frequency of the fundamental mode of vibration of the string on its wavelength:

\[ f_1 = \frac{v}{\lambda_1} \]
Relate the length of the string to the wavelength of the fundamental mode:

\[ \lambda_1 = 2L \]

Substitute for \( \lambda_1 \) to obtain:

\[ f_1 = \frac{v}{2L} \]

Express the dependence of the speed of waves on the string on the tension in the string:

\[ v = \sqrt{\frac{F_1}{\mu}} \]

Substitute for \( v \) in the expression for \( f_1 \) to obtain:

\[ f_1 = \frac{1}{2L} \sqrt{\frac{F_1}{\mu}} \]

(a) Doubling the tension and the length would increase the frequency by a factor of \( \sqrt{2}/2 \).

(b) Halving the tension and keeping the length and the mass per unit length fixed would decrease the frequency by a factor of \( 1/\sqrt{2} \).

(c) Keeping the tension and the mass per unit length fixed and doubling the length will have the fundamental frequency. \( (c) \) is correct.

(d) Keeping the tension and the mass per unit length fixed and halving the length would double the frequency.

13 •• [SSM] Explain how you might use the resonance frequencies of an organ pipe to estimate the temperature of the air in the pipe.

**Determine the Concept** You could measure the lowest resonant frequency \( f \) and the length \( L \) of the pipe. Assuming the end corrections are negligible, the wavelength equals \( 4L \) if the pipe is stopped at one end, and is \( 2L \) if the pipe is open at both ends. Then use \( v = f \lambda \) to find the speed of sound at the ambient temperature. Finally, use \( v = \sqrt{\gamma RT/M} \) (Equation 15-5), where \( \gamma = 1.4 \) for a diatomic gas such as air, \( M \) is the molar mass of air, \( R \) is the universal gas constant, and \( T \) is the absolute temperature, to estimate the temperature of the air.
14  ••  In the fundamental standing-wave pattern of a organ pipe stopped at one end, what happens to the wavelength, frequency, and speed of the sound needed to create the pattern if the air in the pipe becomes significantly colder? Explain your reasoning.

Determine the Concept  The pipe will contract as the air in it becomes significantly colder, and so the wavelength (equal to $4L$) will decrease as well. This effect, however, is negligible compared to the decrease in the speed of sound (recall that the speed of sound in a gas depends on the square root of the absolute temperature). Because $v = f\lambda$ and $v$ decreases with $\lambda$ remaining approximately constant, $f$ must decrease.

15  ••  [SSM]  (a) When a guitar string is vibrating in its fundamental mode, is the wavelength of the sound it produces in air typically the same as the wavelength of the standing wave on the string? Explain. (b) When an organ pipe is in any one of its standing wave modes, is the wavelength of the traveling sound wave it produces in air typically the same as the wavelength of the standing sound wave in the pipe? Explain.

Determine the Concept

(a) No; the wavelength of a wave is related to its frequency and speed of propagation ($\lambda = \frac{v}{f}$). The frequency of the plucked string will be the same as the frequency of the wave it produces in air, but the speeds of the waves depend on the media in which they are propagating. Because the velocities of propagation differ, the wavelengths will not be the same.

(b) Yes. Because both the standing waves in the pipe and the traveling waves have the same speed and frequency, they must have the same wavelength.

16  ••  Figure 16-30 is a photograph of two pieces of very finely woven silk placed one on top of the other. Where the pieces overlap, a series of light and dark lines are seen. This moiré pattern can also be seen when a scanner is used to copy photos from a book or newspaper. What causes the moiré pattern, and how is it similar to the phenomenon of interference?

Determine the Concept  The light is being projected up from underneath the silk, so you will see light where there is a gap and darkness where two threads overlap. Because the two weaves have almost the same spatial period but not exactly identical (because the two are stretched unequally), there will be places where, for large sections of the cloth, the two weaves overlap in phase, leading to brightness, and large sections where the two overlap 90° out of phase (that is, thread on gap and vice versa) leading to darkness. This is exactly the same idea as in the interference of two waves.
17 ** When a musical instrument consisting of drinking glasses, each partially filled to a different height with water, is struck with a small mallet, each glass produces a different frequency of sound wave. Explain how this instrument works.

**Determine the Concept** Standing sound waves are produced in the air columns above the water. The resonance frequency of the air columns depends on the length of the air column, which depends on how much water is in the glass.

18 ** During an organ recital, the air compressor that drives the organ pipes suddenly fails. An enterprising physics student in the audience tries to help by replacing the compressor with a tank of a pressurized tank of nitrogen gas. What effect, if any, will the nitrogen gas have on the frequency output of the organ pipes? What effect, if any, would helium gas have on the frequency output of the organ pipes?

**Picture the Problem** We can use \( v = f \lambda \) to relate the frequency of the sound waves in the organ pipes to the speed of sound in air, nitrogen, and helium. We can use \( v = \sqrt{\gamma RT/M} \) to relate the speed of sound, and hence its frequency, to the properties of the three gases.

Express the frequency of a given note as a function of its wavelength and the speed of sound:

\[
f = \frac{v}{\lambda}
\]

Relate the speed of sound to the absolute temperature and the molar mass of the gas used in the organ:

\[
v = \sqrt{\frac{\gamma RT}{M}}
\]

where \( \gamma \) depends on the kind of gas, \( R \) is a constant, \( T \) is the absolute temperature, and \( M \) is the molar mass.

Substitute for \( v \) to obtain:

\[
f = \frac{1}{\lambda} \sqrt{\frac{\gamma RT}{M}}
\]

For air in the organ pipes we have:

\[
f_{\text{air}} = \frac{1}{\lambda} \sqrt{\frac{\gamma_{\text{air}} RT}{M_{\text{air}}}} \tag{1}
\]

When nitrogen is in the organ pipes:

\[
f_{N_2} = \frac{1}{\lambda} \sqrt{\frac{\gamma_{N_2} RT}{M_{N_2}}} \tag{2}
\]
Express the ratio of equation (2) to equation (1) and solve for $f_{N_2}$:

$$\frac{f_{N_2}}{f_{air}} = \sqrt{\frac{\gamma_{N_2} M_{air}}{\gamma_{air} M_{N_2}}}$$

and

$$f_{N_2} = f_{air} \sqrt{\frac{\gamma_{N_2} M_{air}}{\gamma_{air} M_{N_2}}}$$

Because $\gamma_{N_2} = \gamma_{air}$ and $M_{air} > M_{N_2}$:

$$f_{N_2} > f_{air}$$

That is, $f$ will increase for each organ pipe.

Note, however, that because $M_{air}$ is not very much greater than $M_{N_2}$, the change in frequency will not be very great.

If helium were used, we’d have:

$$f_{He} = f_{air} \sqrt{\frac{\gamma_{He} M_{air}}{\gamma_{air} M_{He}}}$$

Because $\gamma_{He} > \gamma_{air}$ and $M_{air} >> M_{He}$, $f_{He} >> f_{air}$ and the effect will be even more pronounced.

19  The constant $\gamma$ for helium (and all monatomic gases) is 1.67. If a man inhales helium and then speaks, his voice has a high-pitch and becomes cartoon-like. Why?

**Determine the Concept** The wavelength is determined mostly by the resonant cavity of the mouth; the frequency of sounds he makes is equal to the wave speed divided by the wavelength. Because $v_{He} > v_{air}$ (see Equation 15-5), the resonance frequency is higher if helium is the gas in the cavity.

**Estimation and Approximation**

20  It is said that a powerful opera singer can hit a high note with sufficient intensity to shatter an empty wine glass by causing the air in it to resonate at the frequency of her voice. Estimate the frequency necessary to obtain a standing wave in an 8.0-cm-high glass. (The 8.0 cm does not include the height of the stem.) Approximately how many octaves above middle C (262 Hz) is this? **Hint: To go up one octave means to double the frequency.**

**Picture the Problem** If you model the wine glass as a half-open (closed-open) cylinder (shown below on its side), then, knowing the speed of sound in air and
the relationship between the height of the wine glass and the wavelength of its fundamental (1st harmonic) frequency, you can find the fundamental frequency with which it resonates using the relationship \( v = f\lambda \). The diagram shows the displacement-amplitude pattern for the 1st harmonic wave pattern. Note that there is a displacement node at the bottom of the wine glass and a displacement antinode at the top of the wine glass. Note further that, in reality, the displacement antinode is a short distance to the right of the open end of the wine glass.

![Diagram](image)

The fundamental frequency is related to the wavelength of the sound in the wine glass according to:

\[
f_i = \frac{v}{\lambda_i}
\]

Using the diagram, determine the wavelength of the fundamental mode:

\[ L = \frac{1}{4} \lambda_i \Rightarrow \lambda_i = 4L \]

Substituting for \( \lambda_i \) yields:

\[ f_i = \frac{v}{4L} \]

Substitute numerical values and evaluate \( f_i \):

\[ f_i = \frac{343 \text{ m/s}}{4(8.0 \text{ cm})} = 1.072 \text{ kHz} = 1.1 \text{ kHz} \]

Because the frequencies 262 Hz, 524 Hz, and 1024 Hz are \( 2^0, 2^1, \) and \( 2^2 \) times 262 Hz, 1.1 kHz is approximately 2 octaves above 262 Hz.

21. Estimate how accurately you can tune a piano string to a tuning fork of known frequency using only your ears, the tuning fork and a wrench. Explain your answer.

**Determine the Concept** If you do not hear beats for the entire time the string and the tuning fork are vibrating, you can be sure that their frequencies, while not exactly the same, are very close. If the sounds of the vibrating string and the tuning fork last for 10 s, it follows that the beat frequency is less than 0.1 Hz. Hence, the frequencies of the vibrating string and the tuning fork are within 0.1 Hz of each other.
The shortest pipes used in organs are about 7.5 cm long. (a) Estimate the fundamental frequency of a pipe this long that is open at both ends. (b) For such a pipe, estimate the harmonic number \( n \) of the highest-frequency harmonic that is within the audible range. (The audible range of human hearing is about 20 to 20,000 Hz.)

**Picture the Problem** We can use \( v = f_1 \lambda_1 \) to express the resonance frequencies in the organ pipes in terms of their wavelengths and \( L = n \frac{\lambda_n}{2}, n = 1, 2, 3, \ldots \) to relate the length of the pipes to the resonance wavelengths.

(a) Relate the fundamental frequency of the pipe to its wavelength and the speed of sound:

\[
f_1 = \frac{v}{\lambda_1}
\]

Express the condition for constructive interference in a pipe that is open at both ends:

\[
L = n \frac{\lambda_n}{2}, \quad n = 1, 2, 3, \ldots \quad (1)
\]

Solve for \( \lambda_1 \):

\[
\lambda_1 = 2L \Rightarrow f_1 = \frac{v}{2L}
\]

Substitute numerical values and evaluate \( f_1 \):

\[
f_1 = \frac{343 \text{ m/s}}{2(7.5 \times 10^{-2} \text{ m})} = 2.29 \text{ kHz} = 2.3 \text{ kHz}
\]

(b) Relate the resonance frequencies of the pipe to their wavelengths and the speed of sound:

\[
f_n = \frac{v}{\lambda_n}
\]

Solve equation (1) for \( \lambda_n \):

\[
\lambda_n = \frac{2L}{n} \Rightarrow f_n = n \frac{v}{2L}
\]

Substitute numerical values to obtain:

\[
f_n = n \frac{343 \text{ m/s}}{2(7.5 \times 10^{-2} \text{ m})} = n(2.29 \text{ kHz})
\]

Set \( f_n = 20 \text{ kHz} \) and evaluate \( n \):

\[
n = \frac{20 \text{ kHz}}{2.29 \text{ kHz}} = 8.7
\]
The eighth harmonic is within the range defined as audible. The ninth harmonic might be heard by a person with very good hearing.

**23** Estimate the resonant frequencies that are in the audible range of human hearing of the human ear canal. Treat the canal as an air column open at one end, stopped at the other end, and with a length of 1.00 in. How many resonant frequencies lie in this range? Human hearing has been found experimentally to be the most sensitive at frequencies of about 3, 9 and 15 kHz. How do these frequencies compare to your calculations?

**Picture the Problem** If you model the human ear canal as an open-stopped column, then, knowing the speed of sound in air and the relationship between the depth of the ear canal and the wavelength of its fundamental (1st harmonic) frequency, you can find the fundamental frequency with which it resonates using the relationship $v = f\lambda$. The diagram shows the displacement-amplitude pattern for the 1st harmonic wave pattern.

\[ L \]

The frequencies at which our model ear will resonate are given by:

\[ f_n = n \frac{v}{4L}, n = 1, 3, 5, \ldots \]

Solving for $n$ yields:

\[ n = \frac{4Lf_n}{v} \]

The approximate upper limit for a human ear is 20 kHz. Setting $f_n$ equal to 20 kHz and assuming that the temperature of the air in the ear canal is 20°C yields:

\[ n = \frac{4 \left( 1.00 \text{ in} \times \frac{2.54 \text{ cm}}{\text{in}} \right) (20 \text{ kHz})}{343 \text{ m/s}} \]

\[ = 5.92 \approx 5 \]

and

3 resonant frequencies lie within the range of normal human hearing.
The frequencies that lie within the range of human hearing correspond to \( n = 1, 3, \text{ and } 5 \) are:

\[
f_1 = \frac{343 \text{ m/s}}{4(2.54 \text{ cm})} \approx 3.38 \text{ kHz},
\]
\[
f_3 = 3 \left( \frac{343 \text{ m/s}}{4(2.54 \text{ cm})} \right) \approx 10.1 \text{ kHz},
\]
and
\[
f_5 = 5 \left( \frac{343 \text{ m/s}}{4(2.54 \text{ cm})} \right) \approx 16.9 \text{ kHz}
\]

The calculated frequencies agree with the observed frequencies to within 14%.

**Superposition and Interference**

24  
Two harmonic waves traveling on a string in the same direction both have a frequency of 100 Hz, a wavelength of 2.0 cm, and an amplitude of 0.020 m. In addition, they overlap each other. What is the amplitude of the resultant wave if the original waves differ in phase by \( \pi/6 \) and \( \pi/3 \)?

**Picture the Problem**  
We can use \( A = 2y_0 \cos \frac{1}{2} \delta \) to find the amplitude of the resultant wave.

\( (a) \) Evaluate the amplitude of the resultant wave when \( \delta = \pi/6 \):  
\[
A = 2y_0 \cos \frac{1}{2} \delta = 2(0.020 \text{ m}) \cos \frac{1}{2} \left( \frac{\pi}{6} \right)
\]
\[
= 3.9 \text{ cm}
\]

\( (b) \) Proceed as in \( (a) \) with \( \delta = \pi/3 \):  
\[
A = 2y_0 \cos \frac{1}{2} \delta = 2(0.020 \text{ m}) \cos \frac{1}{2} \left( \frac{\pi}{3} \right)
\]
\[
= 3.5 \text{ cm}
\]

25  
[SSM]  
Two harmonic waves having the same frequency, wave speed and amplitude are traveling in the same direction and in the same propagating medium. In addition, they overlap each other. If they differ in phase by \( \pi/2 \) and each has an amplitude of 0.050 m, what is the amplitude of the resultant wave?

**Picture the Problem**  
We can use \( A = 2y_0 \cos \frac{1}{2} \delta \) to find the amplitude of the resultant wave.

Evaluate the amplitude of the resultant wave when \( \delta = \pi/2 \):  
\[
A = 2y_0 \cos \frac{1}{2} \delta = 2(0.050 \text{ m}) \cos \frac{1}{2} \left( \frac{\pi}{2} \right)
\]
\[
= 7.1 \text{ cm}
\]
Two audio speakers facing in the same direction oscillate in phase at the same frequency. They are separated by a distance equal to one-third of a wavelength. Point $P$ is in front of both speakers, on the line that passes through their centers. The amplitude of the sound at $P$ due to either speaker acting alone is $A$. What is the amplitude (in terms of $A$) of the resultant wave at that point?

**Picture the Problem** The phase shift in the waves generated by these two sources is due to their separation of $\lambda/3$. We can find the phase difference due to the path difference from $\delta = 2\pi \frac{\Delta x}{\lambda}$ and then the amplitude of the resultant wave using $A = 2y_0 \cos \frac{1}{2} \delta$.

Evaluate the phase difference $\delta$: $\delta = 2\pi \frac{\Delta x}{\lambda} = 2\pi \frac{\lambda/3}{\lambda} = \frac{2}{3} \pi$

Find the amplitude of the resultant wave: $A_{\text{res}} = 2y_0 \cos \frac{1}{2} \delta = 2A \cos \left( \frac{2}{3} \pi \right)$

$= 2A \cos \frac{\pi}{3} = \frac{A}{2}$

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Two compact sources of sound oscillate in phase with a frequency of 100 Hz. At a point 5.00 m from one source and 5.85 m from the other, the amplitude of the sound from each source separately is $A$. (a) What is the phase difference of the two waves at that point? (b) What is the amplitude (in terms of $A$) of the resultant wave at that point?

**Picture the Problem** The phase shift in the waves generated by these two sources is due to a path difference $\Delta x = 5.85$ m $- 5.00$ m $= 0.85$ m. We can find the phase difference due to this path difference from $\delta = 2\pi \frac{\Delta x}{\lambda}$ and then the amplitude of the resultant wave using $A = 2y_0 \cos \frac{1}{2} \delta$.

(a) Find the phase difference due to the path difference: $\delta = 2\pi \frac{\Delta x}{\lambda}$

Use $v = f\lambda$ to eliminate $\lambda$: $\delta = 2\pi \frac{\Delta x}{v} = 2\pi f \frac{\Delta x}{v}$

Substitute numerical values and evaluate $\delta$: $\delta = 2\pi \left( 100 \text{s}^{-1} \right) \frac{0.85 \text{m}}{343 \text{m/s}} = 1.557 \text{ rad}$

$= 89^\circ$
(b) Relate the amplitude of the resultant wave to the amplitudes of the interfering waves and the phase difference between them:

$$A = 2y_0 \cos \frac{\delta}{2} = 2A \cos \frac{\delta}{2}(1.557)$$

$$= 1.4A$$

With a drawing program or a compass, draw circular arcs of radius 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm and 7 cm centered at each of two points ($P_1$ and $P_2$) a distance $d = 3.0$ cm apart. Draw smooth curves through the intersections corresponding to points $N$ centimeters farther from $P_1$ than from $P_2$ for $N = +2$, +1, 0, –1 and –2, and label each curve with the corresponding value of $N$. There are two additional such curves you can draw, one for $N = +3$ and one for $N = –3$. If identical sources of coherent in-phase 1.0-cm wavelength waves were placed at points $P_1$ and $P_2$, the waves would interfere constructively along each of the smooth curves.

**Picture the Problem** The following diagram was constructed using a spreadsheet program.

29 • [SSM] Two speakers separated by some distance emit sound waves of the same frequency. At some point $P$, the intensity due to each speaker separately is $I_0$. The distance from $P$ to one of the speakers is $\frac{1}{2} \lambda$ longer than that from $P$ to the other speaker. What is the intensity at $P$ if (a) the speakers are coherent and in phase, (b) the speakers are incoherent, and (c) the speakers are coherent and 180° out of phase?

**Picture the Problem** The intensity at the point of interest is dependent on whether the speakers are coherent and on the total phase difference in the waves arriving at the given point. We can use $\delta = 2\pi \frac{\Delta x}{\lambda}$ to determine the phase
difference $\delta$. $A = |2p_0 \cos \frac{1}{2} \delta|$ to find the amplitude of the resultant wave, and the fact that the intensity $I$ is proportional to the square of the amplitude to find the intensity at $P$ for the given conditions.

(a) Find the phase difference $\delta$:

$$\delta = 2\pi \frac{\frac{1}{2} \Delta x}{\lambda} = \pi$$

Find the amplitude of the resultant wave:

$$A = |2p_0 \cos \frac{1}{2} \pi| = 0$$

Because the intensity is proportional to $A^2$:

$$I = 0$$

(b) The sources are incoherent and the intensities add:

$$I = 2I_0$$

(c) The total phase difference is the sum of the phase difference of the sources and the phase difference due to the path difference:

$$\delta_{\text{tot}} = \delta_{\text{sources}} + \delta_{\text{path difference}} = \pi + 2\pi \Delta x \frac{\Delta x}{\lambda} = \pi + 2\pi \left( \frac{1}{2} \right)$$

$$= 2\pi$$

Find the amplitude of the resultant wave:

$$A = |2p_0 \cos \frac{1}{2} (2\pi)| = 2p_0$$

Because the intensity is proportional to $A^2$:

$$I = \frac{A^2}{p_0^2} I_0 = \frac{(2p_0)^2}{p_0^2} I_0 = 4I_0$$

30 • Two speakers separated by some distance emit sound waves of the same frequency. At some point $P'$, the intensity due to each speaker separately is $I_0$. The distance from $P'$ to one of the speakers is one wavelength longer than that from $P'$ to the other speaker. What is the intensity at $P$ if (a) the speakers are coherent and in phase, (b) the speakers are incoherent, and (c) the speakers are coherent and out of phase?
**Picture the Problem** The intensity at the point of interest is dependent on whether the speakers are coherent and on the total phase difference in the waves arriving at the given point. We can use $\delta = 2\pi \frac{\Delta x}{\lambda}$ to determine the phase difference $\delta$, $A = \left| 2p_0 \cos \frac{1}{2} \delta \right|$ to find the amplitude of the resultant wave, and the fact that the intensity is proportional to the square of the amplitude to find the intensity at $P$ for the given conditions.

(a) Find the phase difference $\delta$:

$$\delta = 2\pi \frac{\lambda}{\lambda} = 2\pi$$

Find the amplitude of the resultant wave:

$$A = \left| 2p_0 \cos \frac{1}{2} (2\pi) \right| = 2p_0$$

Because the intensity is proportional to $A^2$:

$$I = \frac{A^2}{p_0^2} I_0 = \frac{(2p_0)^2}{p_0^2} I_0 = 4I_0$$

(b) The sources are incoherent and the intensities add:

$$I = 2I_0$$

(c) The total phase difference is the sum of the phase difference of the sources and the phase difference due to the path difference:

$$\delta_{\text{tot}} = \delta_{\text{sources}} + \delta_{\text{path difference}}$$

$$= \pi + 2\pi \frac{\Delta x}{\lambda} = \pi + 2\pi \left( \frac{\lambda}{\lambda} \right)$$

$$= 3\pi$$

Find the amplitude of the resultant wave:

$$A = \left| 2p_0 \cos \frac{1}{2} (3\pi) \right| = 0$$

Because the intensity is proportional to $A^2$:

$$I = 0$$

31. A transverse harmonic wave with a frequency equal to 40.0 Hz propagates along a taut string. Two points 5.00 cm apart are out of phase by $\pi/6$. (a) What is the wavelength of the wave? (b) At a given point on the string, how much does the phase change in 5.00 ms? (c) What is the wave speed?
**Picture the Problem** (a) Let the +x direction be the direction of propagation of the wave. We can express the phase difference in terms of the separation of the two points and the wavelength of the wave and solve for $\lambda$. (b) We can find the phase difference by relating the time between displacements to the period of the wave. (c) We can use the relationship between the speed, frequency, and wavelength of a wave to find its velocity.

(a) Relate the phase difference to the wavelength of the wave:

$$\delta = 2\pi \frac{\Delta x}{\lambda}$$

Solve for and evaluate $\lambda$:

$$\lambda = 2\pi \frac{\Delta x}{\delta} = 2\pi \frac{5.00\text{ cm}}{\frac{1}{5}\pi} = 60.0\text{ cm}$$

(b) The period of the wave is given by:

$$T = \frac{1}{f} = \frac{1}{40.0\text{ s}^{-1}} = 25.0\text{ ms}$$

Relate the time between the two displacements to the period of the wave:

$$5.00\text{ ms} = \frac{1}{5} T$$

The phase difference corresponding to one-fifth of a period is:

$$\delta = \frac{2\pi}{5}$$

(c) The wave speed is the product of its frequency and wavelength:

$$v = f\lambda = (40.0\text{ s}^{-1})(0.600\text{ m}) = 24.0\text{ m/s}$$

It is thought that the brain determines the direction of the source of a sound by sensing the phase difference between the sound waves striking the eardrums. A distant source emits sound of frequency 680 Hz. When you are directly facing a sound source there is no phase difference. Estimate the phase difference between the sounds received by your ears when you are facing 90° away from the direction of the source.

**Picture the Problem** Assume a distance of about 20 cm between your ears. When you rotate your head through 90°, you introduce a path difference of 20 cm. We can apply the equation for the phase difference due to a path difference to determine the change in phase between the sounds received by your ears as you rotate your head through 90°.
Express the phase difference due to the rotation of your head through $90^\circ$:

$$\delta = 2\pi \frac{20\text{cm}}{\lambda}$$

Because $\lambda = \frac{v}{f}$:

$$\delta = 2\pi \frac{20\text{cm}}{v}$$

Substitute numerical values and evaluate $\delta$:

$$\delta = 2\pi \left( \frac{680\text{ s}^{-1}}{343\text{ m/s}} \right) \frac{20\text{cm}}{343\text{ m/s}} = 0.79\pi \text{ rad}$$

**33  [SSM]** Sound source A is located at $x = 0$, $y = 0$, and sound source B is located at $x = 0$, $y = 2.4$ m. The two sources radiate coherently and in phase. An observer at $x = 15$ m, $y = 0$ notes that as he takes a few steps from $y = 0$ in either the $+y$ or $-y$ direction, the sound intensity diminishes. What is the lowest frequency and the next to lowest frequency of the sources that can account for that observation?

**Picture the Problem** Because the sound intensity diminishes as the observer moves, parallel to a line through the sources, away from his initial position, we can conclude that his initial position is one at which there is constructive interference of the sound coming from the two sources. We can apply the condition for constructive interference to relate the wavelength of the sound to the path difference at his initial position and the relationship between the velocity, frequency, and wavelength of the waves to express this path difference in terms of the frequency of the sources.

Express the condition for constructive interference at $(15 \text{ m}, 0)$:

$$\Delta r = n\lambda, \quad n = 1, 2, 3, \ldots \quad (1)$$

The path difference $\Delta r$ is given by:

$$\Delta r = r_B - r_A$$

Using the Pythagorean theorem, express $r_B$:

$$r_B = \sqrt{(15\text{ m})^2 + (2.4\text{ m})^2}$$

Substitute for $r_B$ to obtain:

$$\Delta r = \sqrt{(15\text{ m})^2 + (2.4\text{ m})^2} - 15\text{ m}$$

Using $v = f\lambda$ and equation (1), express $f_n$ in terms of $\Delta r$ and $n$:

$$f_n = n \frac{v}{\Delta r}, \quad n = 1, 2, 3, \ldots$$
Substituting numerical values yields:

\[ f_n = n \frac{343 \text{ m/s}}{\sqrt{(15 \text{ m})^2 + (2.4 \text{ m})^2} - 15 \text{ m}} = (1.798 \text{ kHz})n = (2 \text{ kHz})n \]

Evaluate \( f_1 \) and \( f_2 \) to obtain:

\[ f_1 = 2 \text{ kHz} \quad \text{and} \quad f_2 = 4 \text{ kHz} \]

**Problem 34** Suppose that the observer in Problem 33 finds himself at a point of minimum intensity at \( x = 15 \text{ m}, y = 0 \). What is then the lowest frequency and the next to lowest frequency of the sources consistent with this observation?

**Picture the Problem** Because the sound intensity increases as the observer moves, parallel to a line through the sources, away from his initial position, we can conclude that his initial position is one at which there is destructive interference of the sound coming from the two sources. We can apply the condition for destructive interference to relate the wavelength of the sound to the path difference at his initial position and the relationship between the velocity, frequency, and wavelength of the waves to express this path difference in terms of the frequency of the sources.

Express the condition for destructive interference at \((15.0 \text{ m}, 0)\):

\[ \Delta r = n \frac{\lambda}{2}, \quad n = 1,3,5,... \quad (1) \]

Express the path difference \( \Delta r \):

\[ \Delta r = r_\text{B} - r_\text{A} \]

Using the Pythagorean theorem, find \( r_\text{B} \):

\[ r_\text{B} = \sqrt{(15 \text{ m})^2 + (2.4 \text{ m})^2} \]

Substitute for \( r_\text{B} \) to obtain:

\[ \Delta r = \sqrt{(15 \text{ m})^2 + (2.4 \text{ m})^2} - 15 \text{ m} \]

Using \( v = f\lambda \) and equation (1), express \( f_n \) in terms of \( \Delta r \) and \( n \):

\[ f_n = n \frac{v}{2\Delta r}, \quad n = 1,3,5,... \]

Substituting numerical values yields:

\[ f_n = n \frac{343 \text{ m/s}}{2\left(\sqrt{(15 \text{ m})^2 + (2.4 \text{ m})^2} - 15 \text{ m}\right)} = (0.8989 \text{ kHz})n = (1 \text{ kHz})n \]

Evaluate \( f_1 \) and \( f_3 \) to obtain:

\[ f_1 = 1 \text{ kHz} \quad \text{and} \quad f_3 = 3 \text{ kHz} \]
Two harmonic water waves of equal amplitudes but different frequencies, wave numbers, and speeds are traveling in the same direction. In addition, they are superposed on each other. The total displacement of the wave can be written as \( y(x,t) = A[\cos(k_1x - \omega_1t) + \cos(k_2x - \omega_2t)] \), where \( \omega_1/k_1 = v_1 \) (the speed of the first wave) and \( \omega_2/k_2 = v_2 \) (the speed of the second wave). (a) Show that \( y(x,t) \) can be written in the form \( y(x,t) = Y(x,t) \cos(k_{av}x - \omega_{av}t) \), where \( \omega_{av} = (\omega_1 + \omega_2)/2 \), \( k_{av} = (k_1 + k_2)/2 \), \( Y(x,t) = 2A \cos\left[\frac{\Delta k}{2}x - \left(\frac{\Delta \omega}{2}\right)t\right] \), \( \Delta \omega = \omega_1 - \omega_2 \), and \( \Delta k = k_1 - k_2 \). The factor \( Y(x,t) \) is called the envelope of the wave. (b) Let \( A = 1.00 \text{ cm}, \omega_1 = 1.00 \text{ rad/s}, k_1 = 1.00 \text{ m}^{-1}, \omega_2 = 0.900 \text{ rad/s}, \) and \( k_2 = 0.800 \text{ m}^{-1} \). Using a spreadsheet program or graphing calculator, make a plot of \( y(x,t) \) versus \( x \) at \( t = 0.00 \text{ s} \) for \( 0 < x < 5.00 \text{ m} \). (c) Using a spreadsheet program or graphing calculator, make three plots of \( Y(x,t) \) versus \( x \) for \( -5.00 \text{ m} < x < 5.00 \text{ m} \) on the same graph. Make one plot for \( t = 0.00 \text{ s} \), the second for \( t = 5.00 \text{ s} \), and the third for \( t = 10.00 \text{ s} \). Estimate the speed at which the envelope moves from the three plots, and compare this estimate with the speed obtained using \( v_{envelope} = \Delta \omega/\Delta k \).

**Picture the Problem** We can use the trigonometric identity

\[
\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)
\]

to derive the expression given in (a) and the speed of the envelope can be found from the second factor in this expression; i.e., from \( \cos\left[\frac{\Delta k}{2}x - \left(\frac{\Delta \omega}{2}\right)t\right] \).

(a) Express the amplitude of the resultant wave function \( y(x,t) \):

\[
y(x,t) = A(\cos(k_1x - \omega_1t) + \cos(k_2x - \omega_2t))
\]

Use the trigonometric identity \( \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \) to obtain:

\[
y(x,t) = 2A \left[ \cos\left(\frac{k_1x - \omega_1t + k_2x - \omega_2t}{2}\right) \cos\left(\frac{k_1x - \omega_1t - k_2x + \omega_2t}{2}\right) \right]
\]

\[
= 2A \left[ \cos\left(\frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{k_1 - k_2}{2} x + \frac{\omega_2 - \omega_1}{2} t\right) \right]
\]
Substitute $\omega_{av} = (\omega_1 + \omega_2)/2$, $k_{av} = (k_1 + k_2)/2$, $\Delta\omega = \omega_1 - \omega_2$ and $\Delta k = k_1 - k_2$ to obtain:

$$y(x,t) = 2A\left[ \cos(k_{av}x - \omega_{av}t) \cos\left( \frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t \right) \right]$$

where

$$Y(x,t) = 2A \cos\left( \frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t \right)$$

(b) A spreadsheet program to calculate $y(x,t)$ between 0 m and 5.00 m at $t = 0.00$ s, follows. The constants and cell formulas used are shown in the table.

<table>
<thead>
<tr>
<th>Cell</th>
<th>Content/Formula</th>
<th>Algebraic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>A15</td>
<td>0</td>
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</tr>
<tr>
<td>A16</td>
<td>A15+0.25</td>
<td>$x + \Delta x$</td>
</tr>
<tr>
<td>B15</td>
<td>2*$B$2<em>COS(0.5</em>($B$3−$B$4)<em>A15−0.5</em>($B$5−$B$6)*$B$8)</td>
<td>$Y(x,0.00)$</td>
</tr>
<tr>
<td>C15</td>
<td>B15<em>COS(0.5</em>($B$3+$B$4)<em>A15−0.5</em>($B$5+$B$6)*$B$7)</td>
<td>$y(x,0.00)$</td>
</tr>
</tbody>
</table>

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<td>32</td>
<td>4.25</td>
<td>1.822</td>
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</table>
A graph of $y(x,0)$ follows:

(c) A spreadsheet program to calculate $Y(x,t)$ for $-5.00 \text{ m} < x < 5.00 \text{ m}$ and $t = 0.00 \text{ s}$, $t = 5.00 \text{ s}$ and $t = 10.00 \text{ s}$ follow: The constants and cell formulas used are shown in the table.

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<tr>
<td>B15</td>
<td>$2<em>BS2</em>COS(0.5*(BS3 -BS4)<em>A15 -0.5</em>(BS5-BS6)*BS7)$</td>
<td>$Y(x,0.00 \text{ s})$</td>
</tr>
<tr>
<td>C15</td>
<td>$2<em>BS2</em>COS(0.5*(BS3 -BS4)<em>A15 -0.5</em>(BS5-BS6)*BS8)$</td>
<td>$Y(x,5.00 \text{ s})$</td>
</tr>
<tr>
<td>D15</td>
<td>$2<em>BS2</em>COS(0.5*(BS3 -BS4)<em>A15 -0.5</em>(BS5-BS6)*BS9)$</td>
<td>$Y(x,10.00 \text{ s})$</td>
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To estimate the speed of the envelope, we can use its horizontal displacement between $t = 0.00$ s and $t = 5.00$ s:

\[ v_{\text{est}} = \frac{\Delta x}{\Delta t} \]  

(1)

From the graph we note that the wave traveled 2.5 m in 5.00 s:

\[ v_{\text{est}} = \frac{2.50 \text{ m}}{5.00 \text{ s}} = 50 \text{ cm/s} \]

The speed of the envelope is given by:

\[ v_{\text{envelope}} = \frac{\Delta \omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \]
Substitute numerical values and evaluate $v_{\text{envelope}}$:

$$v_{\text{envelope}} = \frac{1.00 \text{ rad/s} - 0.900 \text{ rad/s}}{1.00 \text{ m}^{-1} - 0.800 \text{ m}^{-1}} = 50 \text{ cm/s}$$

in agreement with our graphical estimate.

36 ** Two coherent point sources are in phase and are separated by a distance $d$. An interference pattern is detected along a line parallel to the line through the sources and a large distance $D$ from the sources, as shown in Figure 16-31. (a) Show that the path difference $\Delta s$ from the two sources to some point on the line at an angle $\theta$ is given, approximately, by $\Delta s \approx d \sin \theta$. Hint: Assume that $D \gg d$, so the lines from the sources to $P$ are approximately parallel (Figure 16-31b). (b) Show that the two waves interfere constructively at $P$ if $\Delta s = m\lambda$, where $m = 0, 1, 2, ...$ (That is, show there is an interference maximum at $P$ if $\Delta s = m\lambda$, where $m = 0, 1, 2, ...$) (c) Show that the distance $y_m$ from the central maximum (at $y = 0$) to the $m$th interference maximum at $P$ is given by $y_m = D \tan \theta_m$, where $d \sin \theta_m = m\lambda$.

**Picture the Problem** The following diagram shows the two sources separated by a distance $d$ and the path difference $\Delta s$ between the two sources and point $P$. Because the lines from the sources to the distant point are approximately parallel, the triangle shown in the diagram is approximately a right triangle and we can use trigonometry to express $\Delta s$ in terms of $d$ and $\theta$. In Part (b), we can use the relationship giving the phase difference due to a path difference to show that the two waves interfere constructively at $P$ if $\Delta s = m\lambda$. In Part (c) we use the geometry of Figure 16-31a to relate $y_m$ to $D$ and $\theta_m$.

**Part (a)**

(a) Using the diagram for Part (a), relate $\Delta s$ to the separation of the sources and the angle $\theta$:

$$\sin \theta \approx \frac{\Delta s}{d} \Rightarrow \Delta s \approx d \sin \theta$$

(b) $\delta$ is related to $\Delta s$ through the proportion:

$$\frac{\delta}{2\pi} = \frac{\Delta s}{\lambda} \Rightarrow \Delta s = \frac{\delta \lambda}{2\pi}$$
There will be constructive interference at \( P \) provided:

\[
\delta = 2\pi n, \quad m = 0, 1, 2, \ldots
\]

Substituting for \( \delta \) and simplifying yields:

\[
\Delta s = \frac{(2\pi n)\lambda}{2\pi} = m\lambda, \quad m = 0, 1, 2, \ldots
\]

(c) Referring to the diagram for Part (c), note that, if \( \theta_m \ll 1 \) (equivalently, \( d << D \)), then \( d\sin \theta_m = m\lambda \) and:

\[
\tan \theta_m = \frac{y_m}{D} \Rightarrow y_m = D\tan \theta_m
\]

37 ** Two sound sources radiating in phase at a frequency of 480 Hz interfere such that maxima are heard at angles of 0° and 23° from a line perpendicular to that joining the two sources. The listener is at a large distance from the line through both sources, and no additional maxima are heard at angles in the range 0° < \( \theta < 23° \). Find the separation \( d \) between the two sources, and any other angles at which intensity maxima will be heard. (Use the result of Problem 36.)

**Picture the Problem** Because a maximum is heard at 0° and the sources are in phase, we can conclude that the path difference is 0. Because the next maximum is heard at 23°, the path difference to that position must be one wavelength. We can use the result of Part (a) of Problem 36 to relate the separation of the sources to the path difference and the angle \( \theta \). We’ll apply the condition for constructive interference to determine the angular locations of other points of maximum intensity in the interference pattern.

Using the result of Part (a) of Problem 36, express the separation of the sources in terms of \( \Delta s \) and \( \theta \):

\[
d = \frac{\Delta s}{\sin \theta}
\]

Because \( \Delta s = \lambda \) and \( v = f\lambda \):

\[
d = \frac{\lambda}{\sin \theta} = \frac{v}{f\sin \theta}
\]

Evaluate \( d \) with \( \Delta s = \lambda \) and \( \theta = 23° \):

\[
d = \frac{343 \text{ m/s}}{(480 \text{ s}^{-1})\sin 23°} = 1.83 \text{ m} = 1.8 \text{ m}
\]

Express the condition for additional intensity maxima:

\[
d\sin \theta_m = m\lambda
\]

where \( m = 1, 2, 3, \ldots \), or

\[
\theta_m = \sin^{-1}\left(\frac{m\lambda}{d}\right)
\]
Evaluate this expression for $m = 2$:

$$\theta_2 = \sin^{-1} \left[ \frac{2(343 \text{ m/s})}{(480 \text{ s}^{-1})(1.83 \text{ m})} \right] = 51^\circ$$

Remarks: It is easy to show that, for $m > 2$, the inverse sine function is undefined and that, therefore, there are no additional relative maxima at angles larger than $51^\circ$.

38  Two loudspeakers are driven in phase by an audio amplifier at a frequency of 600 Hz. The speakers are on the $y$ axis, one at $y = +1.00 \text{ m}$ and the other at $y = -1.00 \text{ m}$. A listener, starting at $(x, y) = (D, 0)$, where $D >> 2.00 \text{ m}$, walks in the $+y$ direction along the line $x = D$. (See Problem 36.) (a) At what angle $\theta$ will she first hear a minimum in the sound intensity? ($\theta$ is the angle between the positive $x$ axis and the line from the origin to the listener.) (b) At what angle will she first hear a maximum in the sound intensity (after $\theta = 0$)? (c) How many maxima can she possibly hear if she keeps walking in the same direction?

Picture the Problem Because the speakers are driven in phase and the path difference is 0 at her initial position, the listener will hear a maximum at $(D, 0)$. As she walks along a line parallel to the $y$ axis she will hear a minimum wherever it is true that the path difference is an odd multiple of a half wavelength. She will hear an intensity maximum wherever the path difference is an integral multiple of a wavelength. We’ll apply the condition for destructive interference in Part (a) to determine the angular location of the first minimum and, in Part (b), the condition for constructive interference find the angle at which she’ll hear the first maximum after the one at $0^\circ$. In Part (c), we can apply the condition for constructive interference to determine the number of maxima she can hear as keeps walking parallel to the $y$ axis.

(a) Express the condition for destructive interference:

$$d \sin \theta_m = m \frac{\lambda}{2}$$

where $m = 1, 3, 5, \ldots$, or

$$\theta_m = \sin^{-1} \left( \frac{m \lambda}{2d} \right) = \sin^{-1} \left( \frac{mv}{2fd} \right)$$

Evaluate this expression for $m = 1$:

$$\theta_1 = \sin^{-1} \left[ \frac{343 \text{ m/s}}{2(600 \text{ s}^{-1})(2.00 \text{ m})} \right] = 8.22^\circ$$
(b) Express the condition for additional intensity maxima:

\[ d \sin \theta_m = m \lambda \]

where \( m = 0, 1, 2, 3, \ldots \), or

\[ \theta_m = \sin^{-1} \left( \frac{m \lambda}{d} \right) = \sin^{-1} \left( \frac{m v}{f d} \right) \]

Evaluate this expression for \( m = 1 \):

\[ \theta_1 = \sin^{-1} \left( \frac{343 \text{ m/s}}{600 \text{ s}^{-1} \times 2 \text{.00 m}} \right) = 16.6^\circ \]

(c) Express the limiting condition on \( \sin \theta \):

\[ \sin \theta_m = \frac{m \lambda}{d} \leq 1 \Rightarrow m \leq \frac{d}{\lambda} = \frac{f d}{v} \]

Substitute numerical values and evaluate \( m \):

\[ m \leq \frac{600 \text{ s}^{-1} \times 2 \text{.00 m}}{343 \text{ m/s}} = 3.50 \]

Because \( m \) must be an integer:

\[ m = 3 \]

39  [SSM] Two sound sources driven in phase by the same amplifier are 2.00 m apart on the \( y \) axis, one at \( y = +1.00 \) m and the other at \( y = -1.00 \) m. At points large distances from the \( y \) axis, constructive interference is heard at at angles with the \( x \) axis of \( \theta_0 = 0.000 \) rad, \( \theta_1 = 0.140 \) rad and \( \theta_2 = 0.283 \) rad, and at no angles in between (see Figure 16-31). (a) What is the wavelength of the sound waves from the sources? (b) What is the frequency of the sources? (c) At what other angles is constructive interference heard? (d) What is the smallest angle for which the sound waves cancel?

**Picture the Problem** (a) Let \( d \) be the separation of the two sound sources. We can express the wavelength of the sound in terms of the \( d \) and either of the angles at which intensity maxima are heard. (b) We can find the frequency of the sources from its relationship to the speed of the waves and their wavelengths. (c) Using the condition for constructive interference, we can find the angles at which intensity maxima are heard. (d) We can use the condition for destructive interference to find the smallest angle for which the sound waves cancel.

(a) Express the condition for constructive interference:

\[ d \sin \theta_m = m \lambda \Rightarrow \lambda = \frac{d \sin \theta_m}{m} \quad (1) \]

where \( m = 0, 1, 2, 3, \ldots \)

Evaluate \( \lambda \) for \( m = 1 \):

\[ \lambda = (2.00 \text{ m}) \sin(0.140 \text{ rad}) = 0.279 \text{ m} \]
(b) The frequency of the sound is given by:

\[ f = \frac{v}{\lambda} = \frac{343 \text{ m/s}}{0.279 \text{ m}} = 1.23 \text{ kHz} \]

(c) Solve equation (1) for \( \theta_m \):

\[ \theta_m = \sin^{-1} \left( \frac{m\lambda}{d} \right) = \sin^{-1} \left[ \frac{m(0.279 \text{ m})}{2.00 \text{ m}} \right] = \sin^{-1} \left[ \frac{(0.1395) m}{2} \right] \]

The table shows the values for \( \theta \) as a function of \( m \):

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \theta_m ) (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.432</td>
</tr>
<tr>
<td>4</td>
<td>0.592</td>
</tr>
<tr>
<td>5</td>
<td>0.772</td>
</tr>
<tr>
<td>6</td>
<td>0.992</td>
</tr>
<tr>
<td>7</td>
<td>1.35</td>
</tr>
<tr>
<td>8</td>
<td>undefined</td>
</tr>
</tbody>
</table>

(d) Express the condition for destructive interference:

\[ d \sin \theta_m = \frac{m \lambda}{2} \]

where \( m = 1, 3, 5, \ldots \)

Solving for \( \theta_m \) yields:

\[ \theta_m = \sin^{-1} \left( \frac{m \lambda}{2d} \right) \]

Evaluate this expression for \( m = 1 \):

\[ \theta_1 = \sin^{-1} \left[ \frac{0.279 \text{ m}}{2(2.00 \text{ m})} \right] = 0.0698 \text{ rad} \]

40 The two sound sources from Problem 39 are now driven 90° out-of-phase, but at the same frequency as in Problem 39. At what angles are constructive and destructive interference heard?

Picture the Problem The total phase shift in the waves arriving at the points of interest is the sum of the phase shift due to the difference in path lengths from the two sources to a given point and the phase shift due to the sources being out of phase by 90°. From Problem 39 we know that \( \lambda = 0.279 \text{ m} \). Using the conditions on the path difference \( \Delta x \) for constructive and destructive interference, we can find the angles at which intensity maxima are heard.
Letting the subscript "pd" denote "path difference" and the subscript "s" the "sources", express the total phase shift $\delta$:

$$\delta = \delta_{pd} + \delta_s = 2\pi \frac{\Delta x}{\lambda} + \frac{\pi}{4}$$

where $\Delta x$ is the path difference between the two sources and the points at which constructive or destructive interference is heard.

Express the condition for constructive interference:

$$\delta = 2\pi \frac{\Delta x}{\lambda} + \frac{\pi}{4} = 2\pi, 4\pi, 6\pi, ...$$

Solve for $\Delta x$ to obtain:

$$\Delta x = \frac{7}{8}\lambda, \frac{15}{8}\lambda, \frac{23}{8}\lambda, ..., = \frac{(8m-1)}{8}\lambda$$

where $m = 1, 2, 3, ...$

Relate $\Delta x$ to $d$ to obtain:

$$\Delta x = \frac{(8m-1)}{8}\lambda = d \sin \theta_c$$

where the "c" denotes constructive interference.

Solving for $\theta_c$ yields:

$$\theta_c = \sin^{-1}\left[\frac{(8m-1)\lambda}{8d}\right], m = 1, 2, 3, ...$$

The table shows the values for $\theta_c$ for $m = 1$ to 5:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\theta_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.01°</td>
</tr>
<tr>
<td>2</td>
<td>15.2°</td>
</tr>
<tr>
<td>3</td>
<td>23.6°</td>
</tr>
<tr>
<td>4</td>
<td>32.7°</td>
</tr>
<tr>
<td>5</td>
<td>42.8°</td>
</tr>
</tbody>
</table>

Express the condition for destructive interference:

$$\delta = 2\pi \frac{\Delta x}{\lambda} + \frac{\pi}{4} = \pi, 3\pi, 5\pi, ...$$

Solve for $\Delta x$ to obtain:

$$\Delta x = \frac{3}{8}\lambda, \frac{11}{8}\lambda, \frac{19}{8}\lambda, ..., = \frac{(8m-5)}{8}\lambda$$

where $m = 1, 2, 3, ...$

Letting "d" denotes destructive interference, relate $\Delta x$ to $d$ to obtain:

$$\Delta x = \frac{(8m-5)}{8}\lambda = d \sin \theta_d$$
Solving for $\theta_d$ yields:

$$\theta_d = \sin^{-1}\left[\frac{(8m - 5)\lambda}{8d}\right], \quad m = 1, 2, 3, \ldots$$

The table shows the values for $\theta_d$ for $m = 1$ to 5:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\theta_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.00°</td>
</tr>
<tr>
<td>2</td>
<td>11.1°</td>
</tr>
<tr>
<td>3</td>
<td>19.3°</td>
</tr>
<tr>
<td>4</td>
<td>28.1°</td>
</tr>
<tr>
<td>5</td>
<td>37.6°</td>
</tr>
</tbody>
</table>

An astronomical radio telescope consists of two antennas separated by a distance of 200 m. Both antennas are tuned to the frequency of 20 MHz. The signals from each antenna are fed into a common amplifier, but one signal first passes through a phase selector that delays its phase by a chosen amount so that the telescope can "look" in different directions (Figure 16-32). When the phase delay is zero, plane radio waves that are incident vertically on the antennas produce signals that add constructively at the amplifier. What should the phase delay be so that signals coming from an angle $\theta = 10^\circ$ with the vertical (in the plane formed by the vertical and the line joining the antennas) will add constructively at the amplifier? Hint: Radio waves travel at $3.00 \times 10^8$ m/s.

**Picture the Problem** We can calculate the required phase shift from the path difference and the wavelength of the radio waves using $\delta = 2\pi \frac{\Delta s}{\lambda}$.

Express the phase delay as a function of the path difference and the wavelength of the radio waves:

$$\delta = 2\pi \frac{\Delta s}{\lambda} \quad (1)$$

Find the wavelength of the radio waves:

$$\lambda = \frac{v}{f} = \frac{2.998 \times 10^8 \text{ m/s}}{20 \times 10^6 \text{ s}^{-1}} = 15.0 \text{ m}$$

Express the path difference for the signals coming from an angle $\theta$ with the vertical:

$$\Delta s = d \sin \theta$$

Substitute numerical values and evaluate $\Delta s$:

$$\Delta s = (200 \text{ m})\sin 10^\circ = 34.73 \text{ m} \times \frac{1\lambda}{15.0 \text{ m}}$$

$$= 2.315\lambda = 2\lambda + 0.315\lambda$$
Substitute in equation (1) and evaluate $\delta$:

$$\delta = 2\pi \frac{0.315\lambda}{\lambda} \approx 2.0 \text{ rad}$$

**Beats**

42 • When two tuning forks are struck simultaneously, 4.0 beats per second are heard. The frequency of one fork is 500 Hz. (a) What are the possible values for the frequency of the other fork? (b) A piece of wax is placed on the 500-Hz fork to lower its frequency slightly. Explain how the measurement of the new beat frequency can be used to determine which of your answers to Part (a) is the correct frequency of the second fork.

**Picture the Problem** The beat frequency is the difference between the frequencies of the two tuning forks. Let $f_1 = 500$ Hz.

(a) Express the relationship between the beat frequency of the frequencies of the two tuning forks:

$$f_2 = f_1 \pm \Delta f = 500 \text{ Hz} \pm 4 \text{ Hz}$$

Solving for $f_2$ yields:

$$f_2 = 504 \text{ Hz or } 496 \text{ Hz}$$

(b) If the beat frequency increases, then $f_2 = 504$ Hz; if it decreases, $f_2 = 496$ Hz.

43 ••• [SSM] A stationary police radar gun emits microwaves at 5.00 GHz. When the gun is aimed at a car, it superimposes the transmitted and reflected waves. Because the frequencies of these two waves differ, beats are generated, with the speed of the car proportional to the beat frequency. The speed of the car, 83 mi/h, appears on the display of the radar gun. Assuming the car is moving along the line-of-sight of the police officer, and using the Doppler shift equations, (a) show that, for a fixed radar gun frequency, the beat frequency is proportional to the speed of the car. *HINT: Car speeds are tiny compared to the speed of light.*

(b) What is the beat frequency in this case? (c) What is the calibration factor for this radar gun? That is, what is the beat frequency generated per mi/h of speed?

**Picture the Problem** The microwaves strike the speeding car at frequency $f_r$. This frequency will be less than $f_s$ if the car is moving away from the radar gun and greater than $f_s$ if the car is moving toward the radar gun. The frequency shift is given by Equation 15-42 (the low-speed, relative to light, approximation). The car then acts as a moving source emitting waves of frequency $f_r$. The radar gun detects waves of frequency $f'_r$ that are either greater than or less than $f_r$ depending on the direction the car is moving. The total frequency shift is the sum of the two frequency shifts.
(a) Express the frequency difference \( \Delta f' \) as the sum of the frequency difference \( \Delta f_i = f_i - f_s \) and the frequency difference \( \Delta f_2 = f_i' - f_r \):

\[
\Delta f = \Delta f_i + \Delta f_2
\]

Using Equation 15-42, substitute for the frequency differences in equation (1):

\[
\Delta f = -\frac{u}{c} f_s - \frac{u}{c} f_r = -\frac{u}{c} (f_s + f_r)
\]

where \( u = u_s \pm u_t = u_r \) is the speed of the source relative to the receiver.

Apply Equation 15-42 to \( \Delta f_i \) to obtain:

\[
\Delta f_i = \frac{f_i - f_s}{f_s} = -\frac{u_r}{c}
\]

where we’ve used the minus sign because we know the frequency difference is a downshift.

Solving for \( f_r \) yields:

\[
f_r = \left(1 - \frac{u_r}{c}\right) f_s
\]

Substitute for \( f_r \) in equation (2) and simplify to obtain:

\[
\Delta f = -\frac{u}{c} \left( f_s + \left(1 - \frac{u_r}{c}\right) f_s \right)
\]

\[
= -\frac{u_r}{c} \left(2 - \frac{u_r}{c}\right) f_s
\]

\[
= -2 \left(\frac{u_r}{c}\right) f_s + \left(\frac{u_r}{c}\right)^2 f_s
\]

Because \( \frac{u_r}{c} \ll 1 \):

\[
\Delta f \approx -2 f_s \frac{u_r}{c} \Rightarrow \Delta f \propto u_r
\]

(b) Substitute numerical values and evaluate \( |\Delta f| \):

\[
|\Delta f| \approx 2 \left(5.00 \times 10^9 \text{ Hz}\right) \frac{83 \text{ mi/h} \times \frac{0.4470 \text{ m/s}}{1 \text{ mi/h}}}{2.998 \times 10^8 \text{ m/s}} = 1.2 \text{ kHz}
\]

(c) The calibration factor is \( \frac{1.24 \text{ kHz}}{83 \text{ mi/h}} = 15 \text{ Hz/mi/h} \).
Standing Waves

44 • A string fixed at both ends is 3.00 m long. It resonates in its second harmonic at a frequency of 60.0 Hz. What is the speed of transverse waves on the string?

**Picture the Problem** The pictorial representation shows the fixed string vibrating in its second harmonic. We can use \( v = f \lambda \) to relate the second-harmonic frequency to the wavelength of the standing wave for the second harmonic.

Relate the speed of transverse waves on the string to their frequency and wavelength:

\[ v = f \lambda \]

Express \( \lambda_2 \) in terms of the length \( L \) of the string:

\[ \lambda_2 = L \]

Substitute for \( \lambda_2 \) and evaluate \( v \):

\[ v = f \lambda_2 \]

\[ v = f \lambda_2 = \left( \frac{1}{2} \right) \left( 60.0 \text{ s}^{-1} \right) (3.00 \text{ m}) = 180 \text{ m/s} \]

45 • A string 3.00 m long and fixed at both ends is vibrating in its third harmonic. The maximum displacement of any point on the string is 4.00 mm. The speed of transverse waves on this string is 50.0 m/s. (a) What are the wavelength and frequency of this standing wave? (b) Write the wave function for this standing wave.

**Picture the Problem** The pictorial representation shows the string fixed at both ends vibrating in its third harmonic. (a) We can find the wavelength of this standing wave from the standing-wave condition for a string fixed at both ends and its frequency from \( v = f_3 \lambda_3 \). (b) We can use the wave function for a standing wave on a string fixed at both ends \( y_n(x,t) = A_n \sin k_n x \cos \omega_n t \) to write the wave function for the standing wave.
(a) Using the standing-wave condition for a string fixed at both ends, relate the length of the string to the wavelength of the harmonic mode in which it is vibrating:

\[ L = n \frac{\lambda}{2}, \quad n = 1, 2, 3, \ldots \Rightarrow \lambda_3 = \frac{2}{3}L \]

Substitute the numerical value of \( L \) and evaluate \( \lambda_3 \):

\[ \lambda_3 = \frac{2}{3}(3.00\text{ m}) = 2.00\text{ m} \]

Express the frequency of the third harmonic in terms of the speed of transverse waves on the string and their wavelength:

\[ f_3 = \frac{v}{\lambda_3} = \frac{50.0\text{ m/s}}{2.00\text{ m}} = 25.0\text{ Hz} \]

(b) Write the equation for a standing wave, fixed at both ends, in its third harmonic:

\[ y_3(x,t) = A_3 \sin k_3 x \cos \omega_3 t \]

Evaluate \( k_3 \):

\[ k_3 = \frac{2\pi}{\lambda_3} = \frac{2\pi}{2.00\text{ m}} = \pi \text{ m}^{-1} \]

Evaluate \( \omega_3 \):

\[ \omega_3 = 2\pi f_3 = 2\pi(25.0\text{ s}^{-1}) = 50.0\text{ }\pi\text{ s}^{-1} \]

Substitute to obtain:

\[ y_3(x,t) = (4.00\text{ mm})\sin(\pi \text{ m}^{-1})x \cos(50.0\pi \text{ s}^{-1})t \]

46  

Calculate the fundamental frequency for an organ pipe, with an effective length equal to 10 m, that is (a) open at both ends and (b) stopped at one end.

**Picture the Problem** The first harmonic displacement-wave pattern in an organ pipe open at both ends and vibrating in its fundamental mode is represented in Part (a) of the diagram. Part (b) of the diagram shows the wave pattern corresponding to the fundamental frequency for a pipe of the same length \( L \) that is stopped at one end. We can relate the wavelength to the frequency of the fundamental modes using \( v = f\lambda \).
(a) Express the dependence of the frequency of the fundamental mode of vibration in the open pipe on its wavelength:

$$f_{1,\text{open}} = \frac{v}{\lambda_{1,\text{open}}}$$

Relate the length of the open pipe to the wavelength of the fundamental mode:

$$\lambda_{1,\text{open}} = 2L$$

Substitute and evaluate $f_{1,\text{open}}$:

$$f_{1,\text{open}} = \frac{v}{2L} = \frac{343 \text{ m/s}}{2(10 \text{ m})} = 17 \text{ Hz}$$

(b) Express the dependence of the frequency of the fundamental mode of vibration in the closed pipe on its wavelength:

$$f_{1,\text{closed}} = \frac{v}{\lambda_{1,\text{closed}}}$$

Relate the length of the closed pipe to the wavelength of the fundamental mode:

$$\lambda_{1,\text{closed}} = 4L$$

Substitute for $\lambda_{1,\text{closed}}$ to obtain:

$$f_{1,\text{closed}} = \frac{v}{4L}$$

Substitute numerical values and evaluate $f_{1,\text{closed}}$:

$$f_{1,\text{closed}} = \frac{343 \text{ m/s}}{4(10 \text{ m})} = 8.6 \text{ Hz}$$

47 • [SSM] A 5.00-g, 1.40-m long flexible wire has a tension of 968 N and is fixed at both ends. (a) Find the speed of transverse waves on the wire. (b) Find the wavelength and frequency of the fundamental. (c) Find the frequencies of the second and third harmonics.

**Picture the Problem** We can find the speed of transverse waves on the wire using $v = \sqrt{\frac{F_t}{\mu}}$ and the wavelengths of any harmonic from $L = n \frac{\lambda_n}{2}$, where $n = 1, 2, 3, \ldots$. We can use $v = f\lambda$ to find the frequency of the fundamental. For a wire fixed at both ends, the higher harmonics are integer multiples of the first harmonic (fundamental).
(a) Relate the speed of transverse waves on the wire to the tension in the wire and its linear density:

\[ v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{F}{m}} \]

Substitute numerical values and evaluate \( v \):

\[ v = \sqrt{\frac{968 \text{ N}}{(0.00500 \text{ kg})/(1.40 \text{ m})}} = 521 \text{ m/s} \]

(b) Using the standing-wave condition for a wire fixed at both ends, relate the length of the wire to the wavelength of the harmonic mode in which it is vibrating:

\[ L = n\frac{\lambda}{2}, \quad n = 1, 2, 3, \ldots \]

Solve for \( \lambda_1 \):

\[ \lambda_1 = 2L = 2(1.40 \text{ m}) = 2.80 \text{ m} \]

Express the frequency of the first harmonic in terms of the speed and wavelength of the waves:

\[ f_1 = \frac{v}{\lambda_1} = \frac{521 \text{ m/s}}{2.80 \text{ m}} = 186 \text{ Hz} \]

(c) Because, for a wire fixed at both ends, the higher harmonics are integer multiples of the first harmonic:

\[ f_2 = 2f_1 = 2(186 \text{ Hz}) = 372 \text{ Hz} \]

and

\[ f_3 = 3f_1 = 3(186 \text{ Hz}) = 558 \text{ Hz} \]

A taut, 4.00-m-long rope has one end fixed and the other end free. (The free end is attached to a long, light string.) The speed of waves on the rope is 20.0 m/s. (a) Find the frequency of the fundamental. (b) Find the second harmonic. (c) Find the third harmonic.

**Picture the Problem** We can use Equation 16-13, \( f_n = n\frac{v}{4L} = nf_1, n = 1, 3, 5, \ldots \), to find the resonance frequencies for a rope that is fixed at one end.

(a) Using the resonance-frequency condition for a rope fixed at one end, relate the resonance frequencies to the speed of the waves and the length of the rope:

\[ f_n = n\frac{v}{4L} = nf_1, \quad n = 1, 3, 5, \ldots \]

Solving for \( f_1 \) yields:

\[ f_1 = \frac{20.0 \text{ m/s}}{4(4.00 \text{ m})} = 1.25 \text{ Hz} \]
Because this rope is fixed at just one end, it does not support a second harmonic.

For the third harmonic, \( n = 3 \):
\[
 f_3 = 3f_1 = 3(1.25 \text{ Hz}) = \boxed{3.75 \text{ Hz}}
\]

A steel piano wire without windings has a fundamental frequency of 200 Hz. When it is wound with copper wire, its linear mass density is doubled. What is its new fundamental frequency, assuming that the tension is unchanged?

**Picture the Problem** We can find the fundamental frequency of the piano wire using the general expression for the resonance frequencies of a wire fixed at both ends, \( f_n = n\frac{v}{2L} = nf_1, n = 1, 2, 3, ..., \) with \( n = 1 \). We can use \( v = \sqrt{F_T/\mu} \) to express the frequencies of the fundamentals of the two wires in terms of their linear densities.

Relate the fundamental frequency of the piano wire to the speed of transverse waves on it and its linear density:

Express the dependence of the speed of transverse waves on the tension and linear density:

Substituting for \( v \) yields:

Doubling the linear density results in a new fundamental frequency \( f' \) given by:

Substitute the numerical value of \( f_1 \) to obtain:

What is the greatest length that an organ pipe can have in order that its fundamental note be in the audible range (20 to 20,000 Hz) if (a) the pipe is stopped at one end and (b) it is open at both ends?
Superposition and Standing Waves

Picture the Problem Because the frequency and wavelength of sound waves are inversely proportional, the greatest length of the organ pipe corresponds to the lowest frequency in the normal hearing range. We can relate wavelengths to the length of the pipes using the expressions for the resonance frequencies for pipes that are open at both ends and stopped at one end.

Find the wavelength of a 20-Hz note: \[ \lambda_{\text{max}} = \frac{v}{f_{\text{lowest}}} = \frac{343 \text{ m/s}}{20 \text{ s}^{-1}} = 17.2 \text{ m} \]

(a) Relate the length \( L \) of a stopped-at-one-end organ pipe to the wavelengths of its standing waves:

\[ L = n \frac{\lambda_{\text{max}}}{4}, \quad n = 1, 3, 5, \ldots \]

For \( n = 1 \):

\[ L = \frac{\lambda_{\text{max}}}{4} = \frac{17.2 \text{ m}}{4} = \boxed{4.3 \text{ m}} \]

(b) Relate the length \( L \) of an open-at-both-ends organ pipe to the wavelengths of its standing waves:

\[ L = n \frac{\lambda_{\text{max}}}{2}, \quad n = 1, 2, 3, \ldots \]

For \( n = 2 \):

\[ L = \frac{\lambda_{\text{max}}}{2} = \frac{17.2 \text{ m}}{2} = \boxed{8.6 \text{ m}} \]

51 •• [SSM] The wave function \( y(x,t) \) for a certain standing wave on a string fixed at both ends is given by \( y(x,t) = 4.20 \sin(0.200x) \cos(300t) \), where \( y \) and \( x \) are in centimeters and \( t \) is in seconds. (a) A standing wave can be considered as the superposition of two traveling waves. What are the wavelength and frequency of the two traveling waves that make up the specified standing wave? (b) What is the speed of these waves on this string? (c) If the string is vibrating in its fourth harmonic, how long is it?

Picture the Problem We can find \( \lambda \) and \( f \) by comparing the given wave function to the general wave function for a string fixed at both ends. The speed of the waves can then be found from \( v = f \lambda \). We can find the length of the string from its fourth harmonic wavelength.

(a) Using the wave function, relate \( k \) and \( \lambda \):

\[ k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k} \]

Substitute numerical values and evaluate \( \lambda \):

\[ \lambda = \frac{2\pi}{0.200 \text{ cm}^{-1}} = 10\pi \text{ cm} = \boxed{31.4 \text{ cm}} \]
Using the wave function, relate \( f \) and \( \omega \):

\[
\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}
\]

Substitute numerical values and evaluate \( f \):

\[
f = \frac{300 \text{ s}^{-1}}{2\pi} = 47.7 \text{ Hz}
\]

(b) The speed of the traveling waves is the ratio of their angular frequency and wave number:

\[
v = \frac{\omega}{k} = \frac{300 \text{ s}^{-1}}{0.200 \text{ cm}^{-1}} = 15.0 \text{ m/s}
\]

(c) Relate the length of the string to the wavelengths of its standing-wave patterns:

\[
L = n \frac{\lambda}{2}, n = 1, 2, 3, \ldots
\]

Solve for \( L \) when \( n = 4 \):

\[
L = 2\lambda_4 = 2(31.4 \text{ cm}) = 62.8 \text{ cm}
\]

52 ** The wave function \( y(x,t) \) for a certain standing wave on a string that is fixed at both ends is given by \( y(x,t) = (0.0500 \text{ m}) \sin(2.50 \text{ m}^{-1} x) \cos(500 \text{ s}^{-1} t) \). A standing wave can be considered as the superposition of two traveling waves. (a) What are the speed and amplitude of the two traveling waves that result in the specified standing wave? (b) What is the distance between successive nodes on the string? (c) What is the shortest possible length of the string?

**Picture the Problem** (a) \( v, \omega \) and \( k \) are related according to \( \omega = kv \). \( \omega \) and \( k \) can be found from the given wave function. (b) In a standing wave pattern, the nodes are separated by one-half wavelength. (c) Because there is a standing wave on the string, the shortest possible length is one-half the wavelength of the waves interfering to produce the standing wave.

(a) The speed of a traveling wave is the ratio of its angular frequency and wave number:

\[
v = \frac{\omega}{k}
\]

Substitute numerical values and evaluate \( v \):

\[
v = \frac{500 \text{ s}^{-1}}{2.50 \text{ m}^{-1}} = 200 \text{ m/s}
\]

Express the amplitude of the standing wave \( A_{SW} \) in terms of the amplitude of the two traveling waves that result in the standing wave:
Substitute for $A_{SW}$ and evaluate $A$:

$$A = \frac{1}{2}(0.0500 \text{ m}) = \boxed{2.50 \text{ cm}}$$

(b) The distance between nodes is half the wavelength:

$$d = \frac{1}{2} \lambda = \frac{1}{2} \left( \frac{v}{f} \right) = \frac{1}{2} \left( \frac{2 \pi v}{\omega} \right) = \frac{\pi v}{\omega}$$

Substitute numerical values and evaluate $d$:

$$d = \frac{\pi(200 \text{ m/s})}{500 \text{ s}^{-1}} = \boxed{1.26 \text{ m}}$$

(c) Because there is a standing wave on the string, the shortest possible length is:

$$L_{\text{min}} = \frac{1}{2} \lambda = \boxed{1.26 \text{ m}}$$

53 •• A 1.20-m-long pipe is stopped at one end. Near the open end, there is a loudspeaker that is driven by an audio oscillator whose frequency can be varied from 10.0 to 5000 Hz. (Neglect any end corrections.) (a) What is the lowest frequency of the oscillator that will produce resonance within the tube? (b) What is the highest frequency that will produce resonance? (c) How many different frequencies of the oscillator will produce resonance?

**Picture the Problem**

(a) The lowest resonant frequency in this closed-at-one-end tube is its fundamental frequency. This frequency is related to its wavelength through $v = f_{\text{min}} \lambda_{\text{max}}$. (b) We can use the relationship between the $n$th harmonic and the fundamental frequency, $f_n = (2n + 1)f_1, n = 1, 2, 3, \ldots$, to find the highest frequency less than or equal to 5000 Hz that will produce resonance.

(a) The wavelengths of the resonant frequencies in a stopped pipe of length $L$ are given by:

$$\lambda_n = \frac{4L}{n}, n = 1, 3, 5, \ldots$$

$\lambda_{\text{max}}$ corresponds to $n = 1$:

$$\lambda_{\text{max}} = 4L$$

Use $v = f_{\text{min}} \lambda_{\text{max}}$ to express $f_{\text{min}}$:

$$f_{\text{min}} = \frac{v}{\lambda_{\text{max}}} = \frac{v}{4L}$$

Substitute numerical values and evaluate $f_{\text{min}}$:

$$f_{\text{min}} = \frac{343 \text{ m/s}}{4(1.20 \text{ m})} = 71.46 \text{ Hz}$$

$$= \boxed{71.5 \text{ Hz}}$$

(b) Express the $n$th harmonic in terms of the fundamental frequency (first harmonic):

$$f_n = nf_1, n = 1, 3, 5, \ldots$$
To find the highest harmonic below 5000 Hz, let \( f_n = 5000 \) Hz:

\[
5000 \text{ Hz} = n_{\text{highest}} (71.46 \text{ Hz})
\]

Solve for \( n \) (an odd integer) to obtain (because there are only odd harmonics for a pipe stopped at one end):

\[
n = 69
\]

Evaluate \( f_{69} \):

\[
f_{69} = 69f_1 = 69(71.46 \text{ Hz}) = 4934 \text{ kHz}
\]

(c) There are 69 odd harmonics higher than the fundamental frequency, so the total number resonant frequencies is 35.

**54** A 460-Hz tuning fork causes resonance in the tube depicted in Figure 16-33 when the length \( L \) of the air column above the water is 18.3 and 55.8 cm.

(a) Find the speed of sound in air. (b) What is the end correction to adjust for the fact that the antinode does not occur exactly at the end of the open tube?

**Picture the Problem** Sound waves of frequency 460 Hz are excited in the tube, whose length \( L \) can be adjusted. Resonance occurs when the effective length of the tube \( L_{\text{eff}} = L + \Delta L \) equals \( \frac{1}{4} \lambda \), \( \frac{3}{4} \lambda \), \( \frac{5}{4} \lambda \), and so on, where \( \lambda \) is the wavelength of the sound. Even though the pressure node is not exactly at the end of the tube, the wavelength can be found from the fact that the distance between water levels for successive resonances is half the wavelength. We can find the speed from \( v = f\lambda \) and the end correction from the fact that, for the fundamental, \( L_{\text{eff}} = \frac{1}{4} \lambda = L_1 + \Delta L \), where \( L_1 \) is the distance from the top of the tube to the location of the first resonance.

(a) Relate the speed of sound in air to its wavelength and the frequency of the tuning fork:

\[
v = f\lambda
\]

Using the fact that nodes are separated by one-half wavelength, find the wavelength of the sound waves:

\[
\lambda = 2(55.8 \text{ cm} - 18.3 \text{ cm}) = 75.0 \text{ cm}
\]

Substitute numerical values and evaluate \( v \):

\[
v = (460 \text{ s}^{-1})(0.750 \text{ m}) = 345 \text{ m/s}
\]
(b) Relate the end correction $\Delta L$ to the wavelength of the sound and effective length of the tube:

\[ L_{\text{eff}} = \frac{1}{4} \lambda = L_1 + \Delta L \Rightarrow \Delta L = \frac{1}{4} \lambda - L_1 \]

Substitute numerical values and evaluate $\Delta L$:

\[ \Delta L = \frac{1}{4} (75.0 \text{ cm}) - 18.3 \text{ cm} = 0.5 \text{ cm} \]

55  [SSM] An organ pipe has a fundamental frequency of 440.0 Hz at 16.00°C. What will be the fundamental frequency of the pipe if the temperature increases to 32.00°C (assuming the length of the pipe remains constant)? Would it be better to construct organ pipes from a material that expands substantially as the temperature increases or, should the pipes be made of material that maintains the same length at all normal temperatures?

**Picture the Problem** We can use $v = f\lambda$ to express the fundamental frequency of the organ pipe in terms of the speed of sound and $v = \sqrt{\frac{\gamma RT}{M}}$ to relate the speed of sound and the fundamental frequency to the absolute temperature.

Express the fundamental frequency of the organ pipe in terms of the speed of sound:

\[ f = \frac{v}{\lambda} \]

Relate the speed of sound to the temperature:

\[ v = \sqrt{\frac{\gamma RT}{M}} \]

where $\gamma$ and $R$ are constants, $M$ is the molar mass, and $T$ is the absolute temperature.

Substitute for $v$ to obtain:

\[ f = \frac{1}{\lambda} \sqrt{\frac{\gamma RT}{M}} \]

Using primed quantities to represent the higher temperature, express the new frequency as a function of $T$:

\[ f' = \frac{1}{\lambda'} \sqrt{\frac{\gamma RT'}{M}} \]

As we have seen, $\lambda$ is proportional to the length of the pipe. For the first question, we assume the length of the pipe does not change, so $\lambda = \lambda'$. Then the ratio of $f'$ to $f$ is:

\[ \frac{f'}{f} = \sqrt{\frac{T'}{T}} \Rightarrow f' = f \sqrt{\frac{T'}{T}} \]
Evaluate $f'$ for $T' = 305\,\text{K}$ and $T = 289\,\text{K}$:

$$f' = f'_{305\text{K}} = f'_{289\text{K}} \sqrt{\frac{305\text{K}}{289\text{K}}}$$

$$= (440.0\,\text{Hz}) \sqrt{\frac{305\text{K}}{289\text{K}}} = 452\,\text{Hz}$$

Ideally, the pipe should expand so that $v/L$, where $L$ is the length of the pipe, is independent of temperature.

56 According to theory, the end correction for a pipe is approximately $\Delta L = 0.3186D$, where $D$ is the pipe diameter. Find the actual length of a pipe open at both ends that will produce a middle C (256 Hz) as its fundamental mode for pipes of diameter $D = 1.00\,\text{cm}$, $10.0\,\text{cm}$, and $30.0\,\text{cm}$.

**Picture the Problem** We can express the wavelength of the fundamental in a pipe open at both ends in terms of the effective length of the pipe using $\lambda = 2L_{\text{eff}} = 2(L + \Delta L)$, where $L$ is the actual length of the pipe and $\lambda = v/f$.

Solving these equations simultaneously will lead us to an expression for $L$ as a function of $D$.

Express the wavelength of the fundamental in a pipe open at both ends in terms of the pipe’s effective length $L_{\text{eff}}$: $\lambda = 2L_{\text{eff}} = 2(L + \Delta L)$

where $L$ is its actual length.

Solve for $L$ to obtain:

$$L = \frac{\lambda}{2} - \Delta L = \frac{\lambda}{2} - 0.3186D$$

Express the wavelength of middle C in terms of its frequency $f$ and the speed of sound $v$:

$$\lambda = \frac{v}{f}$$

Substitute for $\lambda$ to obtain:

$$L = \frac{v}{2f} - 0.3186D$$

Substitute numerical values to express $L$ as a function of $D$:

$$L = \frac{343\,\text{m/s}}{2(256\,\text{s}^{-1})} - 0.3186D$$

$$= 0.670\,\text{m} - 0.3186D$$

Evaluate $L$ for $D = 1.00\,\text{cm}$:

$$L = 0.670\,\text{m} - 0.3186(0.0100\,\text{m})$$

$$= 66.7\,\text{cm}$$
Evaluate $L$ for $D = 10.0$ cm:

$$L = 0.670\text{m} - 0.3186(0.100\text{m})$$

$$= 63.8\text{cm}$$

Evaluate $L$ for $D = 30.0$ cm:

$$L = 0.670\text{m} - 0.3186(0.300\text{m})$$

$$= 57.4\text{cm}$$

57  •• Assume a 40.0-cm-long violin string has a mass of 1.20 g and is vibrating in its fundamental mode at a frequency of 500 Hz. (a) What is the wavelength of the standing wave on the string? (b) What is the tension in the string? (c) Where should you place your finger to increase the fundamental frequency to 650 Hz?

**Picture the Problem**

(a) We know that, when a string is vibrating in its fundamental mode, its ends are one-half wavelength apart. (b) We can use $v = f\lambda$ to express the fundamental frequency of the violin string in terms of the speed of waves in the string and $v = \sqrt{F_i/\mu}$ to relate the speed of waves in the string and the fundamental frequency to the tension in the string. (c) We can use this relationship between $f$ and $L$, the length of the string, to find the length of string when it vibrates with a frequency of 650 Hz.

(a) Express the wavelength of the standing wave, vibrating in its fundamental mode, to the length $L$ of the string:

$$\lambda = 2L = 2(40.0\text{cm}) = 80.0\text{cm}$$

(b) Relate the speed of the waves combining to form the standing wave to its frequency and wavelength:

$$v = f\lambda$$

Express the speed of transverse waves as a function of the tension in the string:

$$v = \sqrt{\frac{F_i}{\mu}} = \sqrt{\frac{F_iL}{m}}$$

where $m$ is the mass of the string and $L$ is its length.

Substitute for $v$ to obtain:

$$\sqrt{\frac{F_iL}{m}} = f\lambda \Rightarrow F_i = f^2\lambda^2 \frac{m}{L}$$
Substitute numerical values and evaluate $F$:

$$F = (500\text{ s}^{-1})^2 (0.800\text{ m})^2 \left(\frac{1.20 \times 10^{-3} \text{ kg}}{0.400\text{ m}}\right)$$

$$= 480\text{ N}$$

(c) Using $v = f\lambda$ and assuming that the string is still vibrating in its fundamental mode, express its frequency in terms of its length:

$$f = \frac{v}{\lambda} = \frac{v}{2L} \Rightarrow L = \frac{v}{2f}$$

Letting primed quantities refer to a second length and frequency, express $L'$ in terms of $f'$:

$$L' = \frac{v}{2f'}$$

Express the ratio of $L'$ to $L$ and solve for $L'$:

$$\frac{L'}{L} = \frac{f}{f'} \Rightarrow L' = \frac{f'}{f} L$$

Evaluate $L_{650\text{ Hz}}$:

$$L'_{650\text{ Hz}} = \frac{500\text{ Hz}}{650\text{ Hz}} L_{500\text{ Hz}}$$

$$= \frac{500\text{ Hz}}{650\text{ Hz}} (40.0\text{ cm}) = 30.77\text{ cm}$$

You should place your finger 9.2 cm from the fixed end of the string.

58  The G string on a violin is 30.0 cm long. When played without fingering, it vibrates in its fundamental mode at a frequency of 196 Hz. The next higher notes on its C-major scale are A (220 Hz), B (247 Hz), C (262 Hz), and D (294 Hz). How far from the end of the string must a finger be placed to play each of these notes?

**Picture the Problem** Let $f'$ represent the frequencies corresponding to the A, B, C, and D notes and $x(f')$ represent the distances from the end of the string that a finger must be placed to play each of these notes. Then, the distances at which the finger must be placed are given by $x(f') = L(f_G) - L(f')$.

Express the distances at which the finger must be placed in terms of the lengths of the G string and the frequencies $f'$ of the A, B, C, and D notes:

$$x(f') = L(f_G) - L(f') \quad (1)$$
Assuming that it vibrates in its fundamental mode, express the frequency of the G string in terms of its length:

\[ f_G = \frac{v}{\lambda_g} = \frac{v}{2L_g} \Rightarrow L_g = \frac{v}{2f_G} \]

Letting primed quantities refer to the string lengths and frequencies of the A, B, C, and D notes, express \( L' \) in terms of \( f' \):

\[ L' = \frac{v}{2f'} \]

Express the ratio of \( L' \) to \( L \) and solve for \( L' \):

\[ \frac{L'}{L_g} = \frac{f_G}{f'} \Rightarrow L' = \frac{f_G}{f'} L_g \]

<table>
<thead>
<tr>
<th>Note</th>
<th>Frequency</th>
<th>( L(f') )</th>
<th>( x(f') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>220</td>
<td>26.73</td>
<td>3.3</td>
</tr>
<tr>
<td>B</td>
<td>247</td>
<td>23.81</td>
<td>2.6</td>
</tr>
<tr>
<td>C</td>
<td>262</td>
<td>22.44</td>
<td>6.7</td>
</tr>
<tr>
<td>D</td>
<td>294</td>
<td>20.00</td>
<td>10</td>
</tr>
</tbody>
</table>

A string that has a linear mass density of \( 4.00 \times 10^{-3} \) kg/m is under a tension of 360 N and is fixed at both ends. One of its resonance frequencies is 375 Hz. The next higher resonance frequency is 450 Hz. (a) What is the fundamental frequency of this string? (b) Which harmonics have the given frequencies? (c) What is the length of the string?

**Picture the Problem** We can use the fact that the resonance frequencies are multiples of the fundamental frequency to find both the fundamental frequency and the harmonic numbers corresponding to 375 Hz and 450 Hz. We can find the length of the string by relating it to the wavelength of the waves on it and the wavelength to the speed and frequency of the waves. The speed of the waves is, in turn, a function of the tension in the string and its linear density, both of which we are given.
(a) Express 375 Hz as an integer multiple of the fundamental frequency of the string:

\[ nf_1 = 375 \text{ Hz} \] (1)

Express 450 Hz as an integer multiple of the fundamental frequency of the string:

\[ (n+1)f_1 = 450 \text{ Hz} \] (2)

Solve equations (1) and (2) simultaneously for \( f_1 \):

\[ f_1 = \frac{75}{1} \text{ Hz} \]

(b) Substitute in equation (1) to obtain:

\[ n = 5 \Rightarrow \text{the harmonics are the 5}\text{th and 6}\text{th}. \]

(c) Express the length of the string as a function of the speed of transverse waves on it and its fundamental frequency:

\[ L = \frac{\lambda}{2} = \frac{v}{2f_1} \]

Express the speed of transverse waves on the string in terms of the tension in the string and its linear density:

\[ v = \sqrt{\frac{F_T}{\mu}} \]

Substitute for \( v \) to obtain:

\[ L = \frac{1}{2f_1} \sqrt{\frac{F_T}{\mu}} \]

Substitute numerical values and evaluate \( L \):

\[ L = \frac{1}{2(75 \text{ s}^{-1})} \sqrt{\frac{360 \text{ N}}{4.00 \times 10^{-3} \text{ kg/m}}} \]

\[ = \frac{2.0 \text{ m}}{} \]

A string fixed at both ends has successive resonances with wavelengths of 0.54 m for the \( n \)th harmonic and 0.48 m for the \((n + 1)\)th harmonic. (a) Which harmonics are these? (b) What is the length of the string?

**Picture the Problem** (a) We can use the fact that the resonance frequencies are multiples of the fundamental frequency and are expressible in terms of the speed of the waves and their wavelengths to find the harmonic numbers corresponding to wavelengths of 0.54 m and 0.48 m. (b) We can find the length of the string by using the standing-wave condition for a string fixed at both ends.
(a) Express the frequency of the \( n \)th harmonic in terms of its wavelength:

\[ nf_1 = \frac{v}{\lambda_n} = \frac{v}{0.54\, \text{m}} \]

Express the frequency of the \((n + 1)\)th harmonic in terms of its wavelength:

\[ (n + 1)f_1 = \frac{v}{\lambda_{n+1}} = \frac{v}{0.48\, \text{m}} \]

Solve these equations simultaneously for \( n \):

\[ n = 8 \Rightarrow \text{the harmonics are the } 8^{\text{th}} \text{ and } 9^{\text{th}}. \]

(b) Using the standing-wave condition, both ends fixed, relate the length of the string to the wavelength of its \( n \)th harmonic:

\[ L = n \frac{\lambda_n}{2}, \quad n = 1, 2, 3, \ldots \]

Evaluate \( L \) for the eighth harmonic:

\[ L = 8 \left( \frac{0.54\, \text{m}}{2} \right) = 2.2\, \text{m} \]

61 ** [SSM] The strings of a violin are tuned to the tones G, D, A, and E, which are separated by a fifth from one another. That is, \( f(D) = 1.5f(G) \), \( f(A) = 1.5f(D) = 440\, \text{Hz} \), and \( f(E) = 1.5f(A) \). The distance between the bridge at the scroll and the bridge over the body, the two fixed points on each string, is 30.0 cm. The tension on the E string is 90.0 N. (a) What is the linear mass density of the E string? (b) To prevent distortion of the instrument over time, it is important that the tension on all strings be the same. Find the linear mass densities of the other strings.

**Picture the Problem** (a) The mass densities of the strings are related to the transverse wave speed and tension through \( v = \sqrt{\frac{F}{\mu}} \). (b) We can use \( v = f\lambda = 2fL \) to relate the frequencies of the violin strings to their lengths and linear densities.

(a) Relate the speed of transverse waves on a string to the tension in the string and solve for the string’s linear density:

\[ v = \sqrt{\frac{F}{\mu}} \Rightarrow \mu = \frac{F}{v^2} \]

Express the dependence of the speed of the transverse waves on their frequency and wavelength:

\[ v = fE\lambda = 2fEL \]
Substituting for \( v \) gives:

\[
\mu_E = \frac{F_{T,E}}{4f_E^2 L^2}
\]

Substitute numerical values and evaluate \( \mu_E \):

\[
\mu_E = \frac{90.0 \text{ N}}{4\left(1.5(440 \text{ s}^{-1})\right)^2 (0.300 \text{ m})^2} = 5.74 \times 10^{-4} \text{ kg/m} = 0.574 \text{ g/m}
\]

\[(b) \text{ Evaluate } \mu_A:\]

\[
\mu_A = \frac{90.0 \text{ N}}{4(440 \text{ s}^{-1})^2 (0.300 \text{ m})^2} = 1.29 \times 10^{-3} \text{ kg/m} = 1.29 \text{ g/m}
\]

Evaluate \( \mu_D \):

\[
\mu_D = \frac{90.0 \text{ N}}{4(293 \text{ s}^{-1})^2 (0.300 \text{ m})^2} = 2.91 \times 10^{-3} \text{ kg/m} = 2.91 \text{ g/m}
\]

Evaluate \( \mu_G \):

\[
\mu_G = \frac{90.0 \text{ N}}{4(195 \text{ s}^{-1})^2 (0.300 \text{ m})^2} = 6.57 \text{ g/m}
\]

62 On a cello, like most other stringed instruments, the positioning of the fingers by the player determines the fundamental frequencies of the strings. Suppose that one of the strings on a cello is tuned to play a middle C (262 Hz) when played at its full length. By what fraction must that string be shortened in order to play a note that is the interval of a third higher (namely, an E, 330 Hz)? How about a fifth higher or a G (392 Hz)?

**Picture the Problem** The speed of a wave on a string is the product of its wavelength and frequency. In this problem, the standing waves are at the fundamental frequency; that is, the only nodes are at the ends of the strings and the wavelength is twice the length of the string. The speed of a wave on a string is determined by the tension in the string and its mass density. Pressing the string against the neck of the cello does not change the tension in the string appreciably and so we can ignore this very small increase in tension in our solution of the problem. Because the length of the string is half the wavelength of the standing wave on it we compare the lengths of the string for the various notes by comparing the wavelengths corresponding to these frequencies.

The wavelength of a standing wave on a string is given by:

\[
\lambda = \frac{v}{f} = \frac{1}{f} \sqrt{\frac{F_T}{\mu}}
\]
Express the wavelength corresponding to middle C:

\[ \lambda_C = \frac{1}{f_C} \sqrt{\frac{F_T}{\mu}} \]  

(1)

Express he wavelength corresponding to an E:

\[ \lambda_E = \frac{1}{f_E} \sqrt{\frac{F_T}{\mu}} \]  

(2)

Express he wavelength corresponding to a G:

\[ \lambda_G = \frac{1}{f_G} \sqrt{\frac{F_T}{\mu}} \]  

(3)

Divide equation (2) by equation (1) and simplify to obtain:

\[ \frac{\lambda_E}{\lambda_C} = \frac{f_C}{f_E} \frac{\sqrt{\frac{F_T}{\mu}}}{\sqrt{\frac{F_T}{\mu}}} = \frac{f_C}{f_E} \]

Substitute numerical values to obtain:

\[ \frac{\lambda_E}{\lambda_C} = \frac{262 \text{ Hz}}{330 \text{ Hz}} = 0.79 \approx \frac{4}{5} \]

Dividing equation (3) by equation (1) and simplifying yields:

\[ \frac{\lambda_G}{\lambda_C} = \frac{f_C}{f_G} \]

Substitute numerical values to obtain:

\[ \frac{\lambda_E}{\lambda_C} = \frac{262 \text{ Hz}}{392 \text{ Hz}} = 0.67 \approx \frac{2}{3} \]

Thus, to play an E, the string is shortened by one-fifth and, to play a G, the string is shortened by one-third.

To tune your violin, you first tune the A string to the correct pitch of 440 Hz and then you bow both it and an adjoining string simultaneously, all the while listening for beats. While bowing the A and E strings, you hear a beat frequency of 3.00 Hz and note that the beat frequency increases as the tension on the E string is increased. (The E string is to be tuned to 660 Hz.) (a) Why are beats produced by these two strings when bowed simultaneously? (b) What is the frequency of the E string vibration when the beat frequency is 3.00 Hz?

**Picture the Problem** (a) and (b) Beat frequencies are heard when the strings are vibrating with slightly different frequencies. To understand the beat frequency heard when the A and E strings are bowed simultaneously, we need to consider the harmonics of both strings. In Part (c) we’ll relate the tension in the string to the frequency of its vibration and set up a proportion involving the frequencies corresponding to the two tensions that we can solve for the tension when the E string is perfectly tuned.
(a) The two sounds produce a beat because the third harmonic of the A string is the same as the second harmonic of the E string, and the original frequency of the E string is slightly greater than 660 Hz. If $f'_E = (660 + \Delta f)\text{Hz}$, a beat of $2\Delta f$ will be heard.

(b) Because $f_{\text{beat}}$ increases with increasing tension, the frequency of the E string is greater than 660 Hz. Thus the frequency of the E string is:

$$f'_E = 660\text{Hz} + \frac{1}{2}(3.00\text{Hz}) = 661.5\text{Hz} = 662\text{Hz}$$

64. A 2.00-m-long string fixed at one end and free at the other (the free end is fastened to the end of a long, light thread) is vibrating in its third harmonic with a maximum amplitude of 3.00 cm and a frequency 100 Hz. (a) Write the wave function for this vibration. (b) Write a function for the kinetic energy of a segment of the string of length $dx$, at a point a distance $x$ from the fixed end, as a function of time $t$. At what times is this kinetic energy maximum? What is the shape of the string at these times? (c) Find the maximum kinetic energy of the string by integrating your expression for Part (b) over the total length of the string.

**Picture the Problem** $\omega_3$ and $k_3$ can be expressed in terms of the given information and then substituted to find the wave function for the third harmonic. We can use the time-derivative of this expression (the transverse speed) to express the kinetic energy of a segment of mass $dm$ and length $dx$ of the string. Integrating this expression will give us the maximum kinetic energy of the string in terms of its mass.

(a) Write the general form of the wave function for the third harmonic:

$$y_3(x,t) = A_3 \sin k_3 x \cos \omega_3 t$$  \hspace{1cm} (1)

$\omega_3$ is given by:

$$\omega_3 = 2\pi f'_3$$

Using the standing-wave condition for a string fixed at one end, relate the length of the string to its third harmonic wavelength:

$$L = 3 \frac{\lambda_3}{4} \Rightarrow \lambda_3 = \frac{4}{3} L$$

$k_3$ is given by:

$$k_3 = \frac{2\pi}{\lambda_3} = \frac{2\pi}{\frac{4}{3} L} = \frac{3\pi}{2L}$$

Substituting for $\omega_3$ and $k_3$ in equation (1) yields:

$$y_3(x,t) = A_3 \sin \frac{3\pi}{2L} x \cos 2\pi f'_3 t$$
Substitute numerical values to obtain:

\[ y_j(x,t) = \left(0.0300\,\text{m}\right)\sin\left(\frac{3\pi}{4} \frac{\text{m}^{-1}}{\text{m}^{-1}} \right) x \cos\left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}} \right) t \]

where \( x \) is measured from the fixed end and \( 0 \leq x \leq 2.00 \,\text{m} \)

(b) Express the kinetic energy of a segment of string of mass \( dm \):

\[ dK = \frac{1}{2} d\nu_j^2 \]

Express the mass of the segment in terms of its length \( dx \) and the linear density of the string:

\[ dm = \mu dx \]

Using our result in (a), evaluate \( \nu_j \):

\[ \nu_j = \frac{\partial}{\partial t} \left[ \left(0.0300\,\text{m}\right)\sin\left(\frac{3\pi}{4} \frac{\text{m}^{-1}}{\text{m}^{-1}} \right) x \cos\left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}} \right) t \right] \]

\[ = -\left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}}\right)(0.03\,\text{m})\sin\left(\frac{3\pi}{4} \frac{\text{m}^{-1}}{\text{m}^{-1}} \right) x \sin\left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}} \right) t \]

\[ = -(6\pi \,\text{m/s})\sin\left(\frac{3\pi}{4} \frac{\text{m}^{-1}}{\text{m}^{-1}} \right) x \sin\left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}} \right) t \]

Substitute in the expression for \( dK \) to obtain:

\[ dK = \frac{1}{2} \left[ (6\pi \,\text{m/s})\sin\left(\frac{3\pi}{4} \frac{\text{m}^{-1}}{\text{m}^{-1}} \right) x \sin\left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}} \right) t \right]^2 \mu dx \]

Express the condition on the time that makes \( dK \) a maximum:

\[ \sin\left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}} \right) t = 1 \]

or

\[ \left(200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}} \right) t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \]

Solve for and evaluate \( t \):

\[ t = \frac{1}{200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}}} \frac{\pi}{2}, \frac{1}{200\pi \frac{\text{s}^{-1}}{\text{s}^{-1}}} \frac{3\pi}{2}, \ldots \]

\[ = 2.50 \,\text{ms}, 7.50 \,\text{ms}, \ldots \]

Because the string’s maximum kinetic energy occurs when \( y(x,t) = 0 \). Thus the string is a straight line.
(c) Integrate $dK$ from (b) over the length of the string to obtain:

$$K_{\text{max}} = \int_0^L \frac{1}{2} \left[ \omega A \sin{kx} \sin{\omega t} \right]^2 \mu dx$$

$$= \frac{1}{2} \mu \omega^2 A^2 \int_0^L \sin^2{kx} dx$$

$$= \frac{1}{2} \mu \omega^2 A^2 \frac{1}{K} \left[ \frac{1}{2} kx - \frac{1}{4} \sin{2kx} \right]_0^L$$

$$= \frac{1}{4} \mu \omega^2 A^2$$

where $m$ is the mass of the string.

Substitute numerical values and evaluate $K_{\text{max}}$:

$$K_{\text{max}} = \frac{1}{4} m (200 \pi \text{s}^{-1})^2 (0.0300 \text{m})^2$$

$$= \left( \frac{88.8 \text{J/kg}}{m} \right) m$$

65 [SSM] A commonly used physics experiment that examines resonances of transverse waves on a string is shown in Figure 16-34. A weight is attached to the end of a string draped over a pulley; the other end of the string is attached to a mechanical oscillator that moves up and down at a frequency $f$ that remains fixed throughout the demonstration. The length $L$ between the oscillator and the pulley is fixed, and the tension is equal to the gravitational force on the weight. For certain values of the tension, the string resonates. Assume the string does not stretch or shrink as the tension is varied. You are in charge of setting up this apparatus for a lecture demonstration. (a) Explain why only certain discrete values of the tension result in standing waves on the string. (b) Do you need to increase or decrease the tension to produce a standing wave with an additional antinode? Explain. (c) Prove your reasoning in Part (b) by showing that the values for the tension $F_{Tn}$ for the $n$th standing-wave mode are given by $F_{Tn} = 4L^2 f^2 \mu / n^2$, and thus that $F_{Tn}$ is inversely proportional to $n^2$. (d) For your particular setup to fit onto the lecture table, you chose $L = 1.00 \text{ m}$, $f = 80.0 \text{ Hz}$, and $\mu = 0.750 \text{ g/m}$. Calculate how much tension is needed to produce each of the first three modes (standing waves) of the string.

**Picture the Problem** (c) and (d) We can equate the expression for the velocity of a wave on a string and the expression for the velocity of a wave in terms of its frequency and wavelength to obtain an expression for the weight that must be suspended from the end of the string in order to produce a given standing wave pattern. By using the condition on the wavelength that must be satisfied at resonance, we can express the weight on the end of the string in terms of $\mu$, $f$, $L$, and an integer $n$ and then evaluate this expression for $n = 1$, 2, and 3 for the first three standing wave patterns.

(a) Because the frequency is fixed, the wavelength depends only on the tension on the string. This is true because the only parameter that can affect the wave speed on the string is the tension on the string. The tension on the string is provided by the weight hanging from its end. Given that the length of the string is fixed, only
certain wavelengths can resonate on the string. Thus, because only certain wavelengths are allowed, only certain wave speeds will work. This, in turn, means that only certain tensions, and therefore weights, will work.

(b) Higher frequency modes on the same length of string results in shorter wavelengths. To accomplish this without changing frequency, you need to reduce the wave speed. This is accomplished by reducing the tension in the string. Because the tension is provided by the weight on the end of the string, you must reduce the weight.

(c) Express the velocity of a wave on the string in terms of the tension $F_T$ in the string and its linear density $\mu$:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{w}{\mu}}$$

where $w$ is the weight of the object suspended from the end of the string.

Express the wave speed in terms of its wavelength $\lambda$ and frequency $f$:

$$v = f\lambda$$

Equate these expressions for $v$ to obtain:

$$f\lambda = \sqrt{\frac{w}{\mu}} \Rightarrow w = \mu f^2 \lambda^2$$

Express the condition on $\lambda$ that corresponds to resonance:

$$\lambda = \frac{2L}{n}, n = 1, 2, 3, ...$$

Substitute to obtain:

$$w_n = \mu f^2 \left( \frac{2L}{n} \right)^2, n = 1, 2, 3, ...$$

or

$$w_n = \frac{4\mu f^2 L^2}{n^2}, n = 1, 2, 3, ...$$

(d) Substitute numerical values for $L$, $f$, and $\mu$ to obtain:

$$w_n = \frac{4(0.750 \text{ g/m})(80.0 \text{ s}^{-1})^2 (1.00 \text{ m})^2}{n^2}$$

$$= \frac{19.20 \text{ N}}{n^2}$$

Evaluate $w_n$ for $n = 1$:

$$w_1 = \frac{19.20 \text{ N}}{(1)^2} = 19.2 \text{ N}$$

Evaluate $w_n$ for $n = 2$:

$$w_2 = \frac{19.20 \text{ N}}{(2)^2} = 4.80 \text{ N}$$
Evaluate $w_n$ for $n = 3$:

$$w_3 = \frac{19.20 \text{ N}}{(3)^2} = 2.13 \text{ N}$$

**Harmonic Analysis**

66 • A guitar string is given a light pluck at its midpoint. A microphone on your computer detects the sound and a program on the computer determines that most of the subsequent sound consists of a 100-Hz tone accompanied by a bit of sound with a 300-Hz tone. What are the two dominant standing-wave modes on the string?

**Picture the Problem** Plucking a string that is fixed at both ends in the middle results in an antinode at the midpoint of the string. Thus the primary modes of vibration will be those that have an antinode at the midpoint of the string. These modes are the odd harmonics and are given by $f_n = n\left(\frac{v}{2L}\right) = nf_1$ where $n = 1, 3, 5, \ldots$

The odd harmonics of a string that is plucked in the middle are given by: $f_n = n\left(\frac{v}{2L}\right) = nf_1$ where $n = 1, 3, 5, \ldots$

From the information given in the problem statement $f_1 = 100$ Hz . Hence the first dominant standing wave mode is:

Noting that 300 Hz = 3(100 Hz), it follows that the second dominant standing wave mode is:

**Wave Packets**

67 • [SSM] A tuning fork with natural frequency $f_0$ begins vibrating at time $t = 0$ and is stopped after a time interval $\Delta t$. The waveform of the sound at some later time is shown (Figure 16-35) as a function of $x$. Let $N$ be an estimate of the number of cycles in this waveform. (a) If $\Delta x$ is the length in space of this wave packet, what is the range in wave numbers $\Delta k$ of the packet? (b) Estimate the average value of the wavelength $\lambda$ in terms of $N$ and $\Delta x$. (c) Estimate the average wave number $k$ in terms of $N$ and $\Delta x$. (d) If $\Delta t$ is the time it takes the wave packet to pass a point in space, what is the range in angular frequencies $\Delta \omega$ of the packet? (e) Express $f_0$ in terms of $N$ and $\Delta t$. (f) The number $N$ is uncertain
by about ±1 cycle. Use Figure 16-35 to explain why. (g) Show that the uncertainty in the wave number due to the uncertainty in \( N \) is \( 2\pi/\Delta x \).

**Picture the Problem** We can approximate the duration of the pulse from the product of the number of cycles in the interval and the period of each cycle and the wavelength from the number of complete wavelengths in \( \Delta x \). We can use its definition to find the average wave number from the average wavelength.

\[
\Delta t \approx NT = \frac{N}{f_0}
\]

\( (a) \) Relate the duration of the pulse to the number of cycles in the interval and the period of each cycle:

\[
\lambda \approx \frac{\Delta x}{N}
\]

\( (b) \) There are about \( N \) complete wavelengths in \( \Delta x \); hence:

\[
k = \frac{2\pi}{\lambda} = \frac{2\pi N}{\Delta x}
\]

\( (d) \) \( N \) is uncertain because the waveform dies out gradually rather than stopping abruptly at some time; hence, where the pulse starts and stops is not well defined.

\[
\Delta k = \frac{2\pi\Delta N}{\Delta x} = \frac{2\pi}{\Delta x}
\]

because \( \Delta N = \pm 1 \).

**General Problems**

68 ** A 35-m-long string has a linear mass density of 0.0085 kg/m and is under a tension of 18 N. Find the frequencies of the lowest four harmonics if \( (a) \) the string is fixed at both ends, and if \( (b) \) the string is fixed at one end and free at the other. (That is, if the free end is attached to a long string of negligible mass.)

**Picture the Problem** We can use \( v = f_n\lambda_n \) to express the resonance frequencies of the string in terms of their wavelengths and \( L = n\frac{\lambda_n}{2}, n = 1, 2, 3, ... \) to relate the length of the string to the resonance wavelengths for a string fixed at both ends. Our strategy for part \( (b) \) will be the same … except that we’ll use the standing-wave condition \( L = n\frac{\lambda_n}{4}, n = 1, 3, 5, ... \) for strings with one end free.
(a) Relate the frequencies of the harmonics to their wavelengths and the speed of transverse waves on the string:

Express the standing-wave condition for a string with both ends fixed:

Substitute for \( \lambda_n \) to obtain:

Express the speed of the transverse waves as a function of the tension in the string:

Substituting for \( v \) in the expression for \( f_n \) yields:

Substitute numerical values and simplify to obtain:

Use this equation to calculate the 1st four harmonics:

(b) Express the standing-wave condition for a string fixed at one end:

The resonance frequencies equation becomes:

\[
\begin{align*}
L &= n \frac{\lambda_n}{2}, n = 1, 2, 3, \ldots \Rightarrow \lambda_n = \frac{2L}{n} \\
L &= n \frac{\lambda_n}{4}, n = 1, 3, 5, \ldots \Rightarrow \lambda_n = \frac{4L}{n} \\
\end{align*}
\]
Calculate the 1st four harmonics:

\[ f_1 = 0.33 \text{Hz} \]
\[ f_3 = 3(0.3287 \text{Hz}) = 0.99 \text{Hz} \]
\[ f_5 = 5(0.3287 \text{Hz}) = 1.6 \text{Hz} \]

and
\[ f_7 = 7(0.3287 \text{Hz}) = 2.3 \text{Hz} \]

69  

Working for a small gold mining company, you stumble across an abandoned mine shaft that, because of decaying wood shoring, looks too dangerous to explore in person. To measure its depth, you employ an audio oscillator of variable frequency. You determine that successive resonances are produced at frequencies of 63.58 and 89.25 Hz. Estimate the depth of the shaft.

**Picture the Problem** We’ll model the shaft as a pipe of length \( L \) with one end open. We can relate the frequencies of the harmonics to their wavelengths and the speed of sound using \( v = f_n \lambda_n \) and the depth of the mine shaft to the resonance wavelengths using the standing-wave condition for a pipe with one end open:

\[ L = n \frac{\lambda_n}{4}, \quad n = 1, 3, 5, ... \]

Relate the frequencies of the harmonics to their wavelengths and the speed of sound:

\[ f_n = \frac{v}{\lambda_n} \]

Express the standing-wave condition for a pipe with one end open:

\[ L = n \frac{\lambda_n}{4}, \quad n = 1, 3, 5, ... \Rightarrow \lambda_n = \frac{4L}{n} \]

Substitute for \( \lambda_n \) to obtain:

\[ f_n = n \frac{v}{4L} \]

For \( f_n = 63.58 \text{Hz} \):

\[ 63.58 \text{Hz} = n \frac{v}{4L} \]

For \( f_{n+2} = 89.25 \text{Hz} \):

\[ 89.25 \text{Hz} = (n + 2) \frac{v}{4L} \]

Divide either of these equations by the other and solve for \( n \) to obtain:

\[ n = 4.95 \approx 5 \]

Substitute in the equation for \( f_n = f_3 = 63.58 \text{Hz} \):

\[ f_5 = \frac{5v}{4L} \]
Solve for and evaluate \( L \):

\[
L = \frac{5v}{4f_s} = \frac{5(343 \text{ m/s})}{4(63.58 \text{ s}^{-1})} = 6.74 \text{ m}
\]

70 A 5.00-m-long string that is fixed at one end and attached to a long string of negligible mass at the other end is vibrating in its fifth harmonic, which has a frequency of 400 Hz. The amplitude of the motion at each antinode is 3.00 cm. (a) What is the wavelength of this wave? (b) What is the wave number? (c) What is the angular frequency? (d) Write the wave function for this standing wave.

**Picture the Problem** We can use the standing-wave condition for a string with one end free to find the wavelength of the 5th harmonic and the definitions of the wave number and angular frequency to calculate these quantities. We can then substitute in the wave function for a wave in the \( n \)th harmonic to find the wave function for this standing wave.

(a) Express the standing-wave condition for a string with one end free:

\[
L = n \frac{\lambda_n}{4}, \quad n = 1, 3, 5, ...
\]

Solve for and evaluate \( \lambda_5 \):

\[
\lambda_5 = \frac{4L}{5} = \frac{4(5.00 \text{ m})}{5} = 4.00 \text{ m}
\]

(b) Use its definition to calculate the wave number:

\[
k_5 = \frac{2\pi}{\lambda_5} = \frac{2\pi}{4.00 \text{ m}} = \frac{\pi}{2} \text{ m}^{-1}
\]

(c) Using its definition, calculate the angular frequency:

\[
\omega_5 = 2\pi f_5 = 2\pi(400 \text{ s}^{-1}) = 800\pi \text{ s}^{-1}
\]

(d) Write the wave function for a standing wave in the \( n \)th harmonic:

\[
y_n(x, t) = A \sin k_n x \cos \omega_n t
\]

Substitute for \( A \), \( k_5 \), and \( \omega_5 \) to obtain:

\[
y_5(x, t) = A \sin(k_5 x) \cos(\omega_5 t) = (0.0300 \text{ m}) \sin \left( \frac{\pi}{2} \text{ m}^{-1} x \right) \cos(800\pi \text{ s}^{-1}) t
\]

71 The wave function for a standing wave on a string is described by \( y(x,t) = (0.020) \sin(4\pi x) \cos(60\pi t) \), where \( y \) and \( x \) are in meters and \( t \) is in seconds. Determine the maximum displacement and maximum speed of a point on the string at (a) \( x = 0.10 \text{ m} \), (b) \( x = 0.25 \text{ m} \), (c) \( x = 0.30 \text{ m} \), and (d) \( x = 0.50 \text{ m} \).
**Picture the Problem** The coefficient of the factor containing the time dependence in the wave function is the maximum displacement of any point on the string. The time derivative of the wave function is the instantaneous speed of any point on the string and the coefficient of the factor containing the time dependence is the maximum speed of any point on the string.

Differentiate the wave function with respect to $t$ to find the speed of any point on the string:

$$v_y = \frac{\partial}{\partial t} \left[ (0.020) \sin 4\pi x \cos 60\pi t \right] = -(0.020)(60\pi) \sin 4\pi x \sin 60\pi t = -1.2\pi \sin 4\pi x \sin 60\pi t$$

(a) Referring to the wave function, express the maximum displacement of the standing wave:

$$y_{\text{max}}(x) = (0.020\, \text{m}) \sin \left( 4\pi \, \text{m}^{-1} \right) x$$  \hspace{0.5cm} (1)

Evaluate equation (1) at $x = 0.10$ m:

$$y_{\text{max}}(0.10\, \text{m}) = (0.020\, \text{m}) \times \sin \left( 4\pi \, \text{m}^{-1} \right) (0.10\, \text{m}) = 1.9\, \text{cm}$$

Referring to the derivative of the wave function with respect to $t$, express the maximum speed of the standing wave:

$$v_{y,\text{max}}(x) = (1.2\pi \, \text{m/s}) \sin \left( 4\pi \, \text{m}^{-1} \right) x$$  \hspace{0.5cm} (2)

Evaluate equation (2) at $x = 0.10$ m:

$$v_{y,\text{max}}(0.10\, \text{m}) = (1.2\pi \, \text{m/s}) \times \sin \left( 4\pi \, \text{m}^{-1} \right) (0.10\, \text{m}) = 3.6\, \text{m/s}$$

(b) Evaluate equation (1) at $x = 0.25$ m:

$$y_{\text{max}}(0.25\, \text{m}) = (0.020\, \text{m}) \times \sin \left( 4\pi \, \text{m}^{-1} \right) (0.25\, \text{m}) = 0$$

Evaluate equation (2) at $x = 0.25$ m:

$$v_{y,\text{max}}(0.25\, \text{m}) = (1.2\pi \, \text{m/s}) \times \sin \left( 4\pi \, \text{m}^{-1} \right) (0.25\, \text{m}) = 0$$
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(c) Evaluate equation (1) at \( x = 0.30 \text{ m} \):

\[
y_{\text{max}}(0.30 \text{ m}) = (0.020 \text{ m}) \times \sin\left(4\pi \text{ m}^{-1}\right)(0.30 \text{ m})
= -1.2 \text{ cm}
\]

Evaluate equation (2) at \( x = 0.30 \text{ m} \):

\[
v_{y,\text{max}}(0.30 \text{ m}) = \left(1.2\pi \text{ m/s}\right) \times \sin\left(4\pi \text{ m}^{-1}\right)(0.30 \text{ m})
= 2.2 \text{ m/s}
\]

(d) Evaluate equation (1) at \( x = 0.50 \text{ m} \):

\[
y_{\text{max}}(0.50 \text{ m}) = (0.020 \text{ m}) \times \sin\left(4\pi \text{ m}^{-1}\right)(0.50 \text{ m})
= 0
\]

Evaluate equation (2) at \( x = 0.50 \text{ m} \):

\[
v_{y,\text{max}}(0.50 \text{ m}) = (1.2\pi \text{ m/s}) \times \sin\left(4\pi \text{ m}^{-1}\right)(0.50 \text{ m})
= 0
\]

72 A 2.5-m-long string that has a mass of 0.10 kg is fixed at both ends and is under tension of 30 N. When the \( n \)th harmonic is excited, there is a node 0.50 m from one end. (a) What is \( n \)? (b) What are the frequencies of the first three harmonics of this string?

**Picture the Problem** In Part (a) we can use the standing-wave condition for a wire fixed at both ends and the fact that nodes are separated by one-half wavelength to find the harmonic number. In Part (b) we can relate the resonance frequencies to their wavelengths and the speed of transverse waves and express the speed of the transverse waves in terms of the tension in the wire and its linear density.

(a) Express the standing-wave condition for a wire fixed at both ends:

\[
L = n \frac{\lambda_n}{2}, \quad n = 1, 2, 3, ...
\]  

(1)

Solve for and evaluate \( \lambda_1 \):

\[
\lambda_1 = 2L = 2(2.5 \text{ m}) = 5.0 \text{ m}
\]

Relate the distance between nodes to the distance of the node closest to one end and solve for \( \lambda_n \):

\[
\frac{1}{2} \lambda_n = 0.50 \text{ m} \Rightarrow \lambda_n = 1.0 \text{ m}
\]
Solving equation (1) for $n$ yields:

$$n = \frac{2L}{\lambda_n}$$

Substitute for $\lambda_n$ and $L$ and evaluate $n$:

$$n = \frac{2(2.5\text{ m})}{1.0\text{ m}} = 5$$

(b) Express the resonance frequencies in terms of the their wavelengths and the speed of transverse waves on the wire:

$$f_n = \frac{v}{\lambda_n} = n \frac{v}{\lambda_1}$$

Relate the speed of transverse waves on the wire to the tension in the wire:

$$v = \sqrt{\frac{F_L}{\mu}}$$

Substitute for $v$ and simplify to obtain:

$$f_n = n \frac{1}{\lambda_1} \sqrt{\frac{F_L L}{m}}$$

$$= n \frac{1}{5.0\text{ m}} \sqrt{\frac{(30\text{ N})(2.5\text{ m})}{0.10\text{ kg}}}$$

$$= n(5.48\text{ Hz})$$

Evaluate $f_n$ for $n = 1, 2, \text{ and } 3$:

$$f_1 = 5.5\text{ Hz}$$

$$f_2 = 2(5.48\text{ Hz}) = 11\text{ Hz}$$

and

$$f_3 = 3(5.48\text{ Hz}) = 16\text{ Hz}$$

An organ pipe is such that its fundamental frequency is 220 Hz. It is placed in an atmosphere of sulfur hexafluoride (SF$_6$) at the same temperature and pressure. The molar mass of air is $29.0 \times 10^{-3} \text{ kg/mol}$ and the molar mass of SF$_6$ is $146 \times 10^{-3} \text{ kg/mol}$. What is the fundamental frequency of the organ pipe when it is in an atmosphere of SF$_6$?

**Picture the Problem** Because sulfur hexafluoride is a heavy gas, we should expect the fundamental frequency (which is a function of the speed of sound) to be lower in an atmosphere of SF$_6$ than it is in air under normal conditions. We can use the relationship between the speed of a sound wave and its wavelength and frequency, together with $v = \sqrt{\gamma RT / M}$, where $M$ is the molar mass, to express the fundamental frequency of the organ pipe in SF$_6$ in terms of its fundamental frequency in air.
The fundamental frequency of the organ pipe in SF$_6$ is given by:

$$f_{1,\text{SF}_6} = \frac{v_{\text{SF}_6}}{\lambda_1} = \frac{\sqrt{\gamma RT}}{M_{\text{SF}_6}} = \sqrt{\frac{\gamma RT}{\lambda_1^2 M_{\text{SF}_6}}}$$

The fundamental frequency of the organ pipe in air is given by:

$$f_{1,\text{air}} = \frac{v_{\text{air}}}{\lambda_1} = \frac{\sqrt{\gamma RT}}{M_{\text{air}}} = \sqrt{\frac{\gamma RT}{\lambda_1^2 M_{\text{air}}}}$$

Divide the first of these equations by the second and simplify to obtain:

$$\frac{f_{1,\text{SF}_6}}{f_{1,\text{air}}} = \sqrt{\frac{\lambda_1^2 M_{\text{air}}}{\lambda_1^2 M_{\text{SF}_6}}} = \sqrt{\frac{M_{\text{air}}}{M_{\text{SF}_6}}}$$

Solving for $f_{1,\text{SF}_6}$ yields:

$$f_{1,\text{SF}_6} = \sqrt{\frac{M_{\text{air}}}{M_{\text{SF}_6}}} f_{1,\text{air}}$$

Substitute numerical values and evaluate $f_{1,\text{SF}_6}$:

$$f_{1,\text{SF}_6} = \sqrt{\frac{29.0 \times 10^{-3} \text{ kg/mol}}{146 \times 10^{-3} \text{ kg/mol}}} (220 \text{ Hz})$$

$$= 98.0 \text{ Hz}$$

During a lecture demonstration of standing waves, one end of a string is attached to a device that vibrates at 60 Hz and produces transverse waves of that frequency on the string. The other end of the string passes over a pulley, and the tension is varied by attaching weights to that end. The string has approximate nodes next to both the vibrating device and the pulley. (a) If the string has a linear mass density of 8.0 g/m and is 2.5 m long (from the vibrating device to the pulley), what must be the tension for the string to vibrate in its fundamental mode? (b) Find the tension necessary for the string to vibrate in its second, third, and fourth harmonic.

**Picture the Problem** We can use $v = \sqrt{\frac{F_T}{\mu}}$ to express $F_T$ as a function of $v$ and $v = f\lambda$ to relate $v$ to the frequency and wavelength of the string’s fundamental mode. Because, for a string fixed at both ends, $f_n = nf_1$, we can extend our result in Part (a) to Part (b).

(a) Relate the speed of the transverse waves on the string to the tension in it:

$$v = \sqrt{\frac{F_T}{\mu}} \Rightarrow F_T = \mu v^2 \quad (1)$$
Relate the speed of the transverse waves on the string to their frequency and wavelength:

\[ v = f_i \lambda_i \]

Express the wavelength of the fundamental mode to the length of the string:

\[ \lambda_i = 2L \]

Substitute for \( \lambda_i \) to obtain:

\[ v = 2fL \]

Substitute for \( v \) in equation (1) to obtain:

\[ F_T = 4f^2L^2\mu \]  \hspace{1cm} (2)

Substitute numerical values and evaluate \( F \):

\[ F_T = 4(60\text{ s}^{-1})^2(2.5\text{ m})^2(8.0 \times 10^{-3}\text{ kg/m}) = \boxed{0.72\text{ kN}} \]

(b) For the \( n \)th harmonic, equation (2) becomes:

\[ F_n = f_n^2L^2\mu = n^2f_i^2L^2\mu = n^2(720\text{ N}) \]

Evaluate this expression for \( n = 2, 3, \) and 4:

\[ F_2 = 4(720\text{ N}) = \boxed{2.9\text{ kN}} \]
\[ F_3 = 9(720\text{ N}) = \boxed{6.5\text{ kN}} \]

and

\[ F_4 = 16(720\text{ N}) = \boxed{12\text{ kN}} \]

Three successive resonance frequencies in an organ pipe are 1310, 1834, and 2358 Hz. (a) Is the pipe closed at one end or open at both ends? (b) What is the fundamental frequency? (c) What is the effective length of the pipe?

**Picture the Problem** (a) We can use the conditions \( \Delta f = f_i \) and \( f_n = nf_i \), where \( n \) is an integer, which must be satisfied if the pipe is open at both ends to decide whether the pipe is closed at one end or open at both ends. (b) Once we have decided this question, we can use the condition relating \( \Delta f \) and the fundamental frequency to determine the latter. In Part (c) we can use the standing-wave condition for the appropriate pipe to relate its length to its resonance wavelengths.

(a) Express the conditions on the frequencies for a pipe that is open at both ends:

\[ \Delta f = f_i \]

and

\[ f_n = nf_i \]
Evaluate $\Delta f = f_1$:
\[ \Delta f = 1834 \text{ Hz} - 1310 \text{ Hz} = 524 \text{ Hz} \]

Using the 2nd condition, find $n$:
\[ n = \frac{f_n}{f_1} = \frac{1310 \text{ Hz}}{524 \text{ Hz}} = 2.5 \]

The pipe is closed at one end.

(b) Express the condition on the frequencies for a pipe that is open at both ends:
\[ \Delta f = 2f_1 \Rightarrow f_1 = \frac{1}{2} \Delta f \]

Substitute numerical values and evaluate $f_1$:
\[ f_1 = \frac{1}{2}(524 \text{ Hz}) = 262 \text{ Hz} \]

(c) Using the standing-wave condition for a pipe open at one end, relate the effective length of the pipe to its resonance wavelengths:
\[ L = n \frac{\lambda_n}{4}, \quad n = 1, 3, 5, ... \]

For $n = 1$ we have:
\[ \lambda_1 = \frac{v}{f_1} \quad \text{and} \quad L = \frac{\lambda_1}{4} = \frac{v}{4f_1} \]

Substitute numerical values and evaluate $L$:
\[ L = \frac{343 \text{ m/s}}{4(262 \text{ s}^{-1})} = 32.7 \text{ cm} \]

76 During an experiment studying the speed of sound in air using an audio oscillator and a tube open at one end and stopped at the other, a particular resonant frequency is found to have nodes roughly 6.94 cm apart. The oscillator’s frequency is increased, and the next resonant frequency found has nodes 5.40 cm apart. (a) What are the two resonant frequencies? (b) What is the fundamental frequency? (c) Which harmonics are these two modes? The speed of sound is 343 m/s.

Picture the Problem (a) Because adjacent nodes are separated by one-half wavelength, we can find the frequencies from our knowledge of the speed of sound in air the wavelengths of the standing-wave patterns. (b) These frequencies are consecutive odd-multiples (the tube is half-open) of the fundamental frequency. (c) The frequencies found in Part (a) are integer multiples of the fundamental frequency.
(a) The frequencies are determined by the speed of sound in air and by the wavelengths of the standing-wave patterns:

\[ f_n = \frac{\nu}{\lambda_n} \]

Because the nodes are a half-wavelength apart:

\[ \lambda_n = 2d_{\text{node-to-node}} \]

Substitute for \( \lambda_n \) to obtain:

\[ f_n = \frac{\nu}{2d_{\text{node-to-node}}} \]

Substitute numerical values and evaluate the two frequencies:

\[
\begin{align*}
 f_n &= \frac{343 \text{ m/s}}{2(6.94 \text{ cm})} = 2471 \text{ Hz} \\
 &= 2.47 \text{ kHz} \\
 f_n' &= \frac{343 \text{ m/s}}{2(5.40 \text{ cm})} = 3176 \text{ Hz} \\
 &= 3.18 \text{ kHz}
\end{align*}
\]

(b) Assuming the two frequencies are adjacent resonant frequencies, they are odd multiples (because the tube is half-open) of the fundamental frequency:

\[
\begin{align*}
 nf_1 &= 2471 \text{ Hz} \\
 (n + 2)f_1 &= 3176 \text{ Hz}
\end{align*}
\]

Subtract the first of these equations from the second to obtain:

\[
\begin{align*}
 (n + 2)f_1 - nf_1 &= 3176 \text{ Hz} - 2471 \text{ Hz} \\
 &= 705 \text{ Hz}
\end{align*}
\]

Solving for \( f_1 \) yields:

\[
\begin{align*}
 f_1 &= \frac{1}{2}(705 \text{ Hz}) = 353 \text{ Hz}
\end{align*}
\]

(c) Divide \( f_n \) and \( f_n' \) by \( f_1 \) to obtain:

\[
\begin{align*}
 n &= \frac{2471 \text{ Hz}}{353 \text{ Hz}} = 7 \\
 n' &= \frac{3176 \text{ Hz}}{353 \text{ Hz}} = 9
\end{align*}
\]

A standing wave on a rope is represented by the wave function

\[
y(x,t) = (0.020 \text{ m}) \sin \left( \frac{1}{2} \pi x \right) \cos (40 \pi t), \text{ where } x \text{ and } y \text{ are in meters and } t \text{ is in seconds. (a) Write wave functions for two traveling waves that, when}
\]
superimposed, will produce this standing-wave pattern. (b) What is the distance between the nodes of the standing wave? (c) What is the maximum speed of the rope at \( x = 1.0 \) m? (d) What is the maximum acceleration of the rope at \( x = 1.0 \) m?

**Picture the Problem** We know that the superimposed traveling waves have the same wave number and angular frequency as the standing-wave function, have equal amplitudes that are half that of the standing-wave function, and travel in opposite directions. From inspection of the standing-wave function we note that \( k = \frac{1}{2} \pi \, \text{m}^{-1} \) and \( \omega = 40 \pi \, \text{s}^{-1} \). We can express the speed of a segment of the rope by differentiating the standing-wave function with respect to time and the acceleration by differentiating the velocity function with respect to time.

(a) Write the wave function for the wave traveling in the +x direction:

\[
y_1(x,t) = (0.010 \, \text{m}) \sin \left[ \frac{\pi}{2} \, \text{m}^{-1} \right] \left[ x - (40 \pi \, \text{s}^{-1})t \right]
\]

Write the wave function for the wave traveling in the −x direction:

\[
y_2(x,t) = (0.010 \, \text{m}) \sin \left[ \frac{\pi}{2} \, \text{m}^{-1} \right] \left[ x + (40 \pi \, \text{s}^{-1})t \right]
\]

(b) Express the distance \( d \) between adjacent nodes in terms of the wavelength of the standing wave:

\[ d = \frac{1}{2} \lambda \]

Use the wave number to find the wavelength:

\[ k = \frac{1}{2} \pi \, \text{m}^{-1} = \frac{2\pi}{\lambda} \text{ and } \lambda = 4.00 \, \text{m} \]

Substitute for \( \lambda \) and evaluate \( d \):

\[ d = \frac{1}{2} (4.00 \, \text{m}) = 2.00 \, \text{m} \]

(c) Differentiate the given wave function with respect to \( t \) to express the speed of any segment of the rope:

\[
v_y(x,t) = \frac{\partial}{\partial t} \left[ (0.020 \, \text{m}) \sin \left( \frac{\pi}{2} \, \text{m}^{-1} \right) x \cos (40 \pi \, \text{s}^{-1})t \right]
\]

\[
= -0.80 \pi \, \text{m/s} \sin \left( \frac{\pi}{2} \, \text{m}^{-1} \right) x \sin (40 \pi \, \text{s}^{-1})t
\]
Evaluate \( v_y(1.0 \text{ m}, t) \):

\[
v_y(1.0 \text{ m}, t) = -(0.80 \pi \text{ m/s})\sin \left( \frac{\pi}{2} \text{ m}^{-1} \right) (1 \text{ m})\sin(40 \pi \text{ s}^{-1} t) \\
= (0.80 \pi \text{ m/s})\sin(40 \pi \text{ s}^{-1} t) \\
= (2.5 \text{ m/s})\sin(40 \pi \text{ s}^{-1} t)
\]

The maximum speed of the rope at \( x = 1.0 \text{ m} \) occurs when
\[
v_{\text{max}} = 2.5 \text{ m/s}
\]

\( (d) \) Differentiate \( v_y(x, t) \) with respect to time to obtain \( a_y(x, t) \):

\[
a_y(x, t) = \frac{\partial}{\partial t} \left[ -(0.80 \pi \text{ m/s})\sin \left( \frac{\pi}{2} \text{ m}^{-1} \right) x\sin(40 \pi \text{ s}^{-1} t) \right] \\
= -(32 \pi^2 \text{ m}^2/\text{s}^2)\sin \left( \frac{\pi}{2} \text{ m}^{-1} \right) x\cos(40 \pi \text{ s}^{-1} t) t
\]

Evaluate \( a_y(1.0 \text{ m}, t) \):

\[
a_y(1.0 \text{ m}, t) = \pm(32 \pi^2 \text{ m}^2/\text{s}^2)\sin \left( \frac{\pi}{2} \text{ m}^{-1} \right) (1 \text{ m})\cos(40 \pi \text{ s}^{-1} t) t \\
= \pm(32 \pi^2 \text{ m}^2/\text{s}^2)\cos(40 \pi \text{ s}^{-1} t) t \\
= \pm(0.32 \text{ km/s}^2)\cos(40 \pi \text{ s}^{-1} t) t
\]

The maximum acceleration of the
\[
v_{\text{max}} = \pm 0.32 \text{ km/s}^2
\]

rope at \( x = 1.0 \text{ m} \) occurs when
\[
\cos(40 \pi \text{ s}^{-1} t) = 1:
\]

Two traveling wave pulses on a string are represented by the wave functions \( y_1(x, t) = \frac{0.020}{2.0 + (x - 2.0 t)^2} \) and \( y_2(x, t) = \frac{-0.020}{2.0 + (x + 2.0 t)^2} \), where \( x \) is in meters and \( t \) is in seconds. \( (a) \) Using a spreadsheet program or graphing calculator, make a graph of each wave function separately as a function of \( x \) at \( t = 0 \) and again at \( t = 1.0 \text{ s} \) and describe the behavior of each as time increases. For each graph make your plot for \(-5.0 < x < +5.0 \text { m} \). \( (b) \) Graph the resultant wave function at \( t = -1.0 \text{ s} \), at \( t = 0.0 \text{ s} \) and at \( t = 1.0 \text{ s} \).
**Picture the Problem** We’ll use a spreadsheet program to graph the wave functions and their sum as functions of $x$ at $t = 0$ and at $t = 1.0$ s. In (b) we can add the wave functions algebraically to find the result wave function at $t = 0$ and at $t = 1.0$ s.

(a) and (b) Part of the spreadsheet program to calculate values for $y_1(x,t)$ and $y_2(x,t)$ in the interval $-5.0 \, m < x < +5.0 \, m$ for the given times follows. The constants and cell formulas used are shown in the table.

<table>
<thead>
<tr>
<th>Cell</th>
<th>Content/Formula</th>
<th>Algebraic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>A5</td>
<td>-5.0</td>
<td>$x$</td>
</tr>
<tr>
<td>A6</td>
<td>A5+0.1</td>
<td>$x + \Delta x$</td>
</tr>
<tr>
<td>B1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>B2</td>
<td>0</td>
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<tr>
<td>B3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>B5</td>
<td>0.02/(2+(A5−2*$B$2)^2)</td>
<td>$y_1(x,0)$</td>
</tr>
<tr>
<td>C5</td>
<td>0.02/(2+(A5−2*$B$3)^2)</td>
<td>$y_1(x,1.0, s)$</td>
</tr>
<tr>
<td>D5</td>
<td>-0.02/(2+(A5+2*$B$2)^2)</td>
<td>$y_2(x,0)$</td>
</tr>
<tr>
<td>E5</td>
<td>-0.02/(2+(A5+2*$B$3)^2)</td>
<td>$y_2(x,1.0, s)$</td>
</tr>
<tr>
<td>F5</td>
<td>0.02/(2+(A5−2*$B$1)^2)−0.02/(2+(A5+2*$B$1)^2)</td>
<td>$y_1(x,−1) + y_2(x,−1)$</td>
</tr>
<tr>
<td>G5</td>
<td>B5+D5</td>
<td>$y_1(x,0) + y_2(x,0)$</td>
</tr>
<tr>
<td>H5</td>
<td>0.02/(2+(A5−2*$B$3)^2)−0.02/(2+(A5+2*$B$3)^2)</td>
<td>$y_1(x,1) + y_2(x,1)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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<tr>
<td>4</td>
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<td>$y_1(x,1 , s)$</td>
<td>$y_2(x,0)$</td>
<td>$y_2(x,1 , s)$</td>
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<td>-0.001</td>
<td>-0.002</td>
</tr>
<tr>
<td>9</td>
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<td>0.000</td>
<td>-0.001</td>
<td>-0.002</td>
</tr>
<tr>
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<td>0.000</td>
<td>-0.001</td>
<td>-0.002</td>
</tr>
<tr>
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<td>0.001</td>
<td>0.002</td>
<td>-0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>111</td>
<td>4.8</td>
<td>0.001</td>
<td>0.002</td>
<td>-0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>112</td>
<td>4.9</td>
<td>0.001</td>
<td>0.002</td>
<td>-0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>113</td>
<td>5.0</td>
<td>0.001</td>
<td>0.002</td>
<td>-0.001</td>
<td>0.000</td>
</tr>
</tbody>
</table>
The following graph of $y_1(x,t)$ shows it traveling from left to right.

The following graph of $y_2(x,t)$ shows it traveling from right to left.

(b) The following graph shows the resultant wave function at $t = -1.0$ s, at $t = 0.0$ s and at $t = 1.0$ s.
Three waves that have the same frequency, wavelength, and amplitude are traveling along the \( x \) axis. The three waves are described by the following wave functions:

\[ y_1(x, t) = (5.00 \text{ cm}) \sin \left( kx - \omega t - \frac{\pi}{3} \right), \]

\[ y_2(x, t) = (5.00 \text{ cm}) \sin (kx - \omega t), \]

\[ y_3(x, t) = (5.00 \text{ cm}) \sin (kx - \omega t + \frac{\pi}{3}), \]

where \( x \) is in meters and \( t \) is in seconds. The resultant wave function is given by

\[ y(x, t) = A \sin (kx - \omega t + \delta). \]

What are the values of \( A \) and \( \delta \)?

**Picture the Problem** A harmonic function can be represented by a vector rotating at the angular frequency \( \omega \) (see Chapter 14). The simplest way to do this problem is to use that representation. The vectors, of equal magnitude, are shown in the diagram. We can find the resultant wave function by finding the magnitude and direction of the resultant vector.

From the diagram it is evident that:

\[ \sum y_y = 0 \]

Find the sum of the \( x \) components of the vectors:

\[ \sum y_x = (5.00 \text{ cm}) \cos 60^\circ + (5.00 \text{ cm}) \cos 60^\circ = 10.0 \text{ cm} \]

Relate the magnitude of the resultant vector to the sum of its \( x \) and \( y \) components:

\[ A = \sqrt{\left( \sum y_x \right)^2 + \left( \sum y_y \right)^2} \]

\[ = \sqrt{(10.0 \text{ cm})^2 + (0)^2} = 10.0 \text{ cm} \]

The direction of the resultant vector is \( \delta \):

\[ \delta = \tan^{-1} \left( \frac{\sum y_y}{\sum y_x} \right) = \tan^{-1} \left( \frac{0}{10.0 \text{ cm}} \right) = 0 \]

A harmonic pressure wave produced by a distant source is traveling through your vicinity, and the wave fronts that travel through your vicinity are vertical planes. Let the \( +x \) direction be to the east and the \( +y \) direction be toward the north. The wave function for the wave is \( p(x, y, t) = A \cos (k_x x + k_y y - \omega t) \).
Show that the direction in which the wave is traveling makes an angle \( \theta = \tan^{-1}(k_y/k_x) \) with the \(+x\) direction and that the wave speed \( v \) is given by
\[
v = \omega \sqrt{k_x^2 + k_y^2}.
\]

**Picture the Problem** The diagram shows a snapshot of a two-dimensional plane wave propagating at an angle \( \theta \) with respect to the \(+x\) axis. The view is along the \(-z\) axis. The wave itself moves in a direction perpendicular to the wavefront. Choose two points \((x_1, y_1)\) and \((x_2, y_2)\) that have a separation of exactly 1 wavelength along the wave propagation direction. Let the snapshot be taken at some fixed time \( t \).

Applying the Pythagorean theorem to the right triangle in the diagram yields:
\[
\lambda = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}
\]  

Because \( k_x \) and \( k_y \) are the \(x\) and \(y\) components of \( \vec{k} \):
\[
k_x = k \cos \theta
\]  

and
\[
k_y = k \sin \theta
\]

Divide the second of these equations by the first to obtain:
\[
\tan \theta = \frac{k_y}{k_x} \Rightarrow \theta = \tan^{-1}\left(\frac{k_y}{k_x}\right)
\]

The wave speed \( v \) is the ratio of the angular frequency \( \omega \) and wave number \( k \):
\[
v = \frac{\omega}{k} = \frac{\lambda}{2\pi}
\]  

(2)
At time $t$, the phase at point 1 is $k_x x_1 + k_y y_1 - \omega t$ and the phase at point 2 is $k_x x_2 + k_y y_2 - \omega t$, so the phase difference is:

$$\delta = k_x x_2 + k_y y_2 - \omega t - (k_x x_1 + k_y y_1 - \omega t) = k_x \Delta x + k_y \Delta y$$

or, because the waves are separated by one wavelength, $\delta = 2\pi$ and we have $k_x \Delta x + k_y \Delta y = 2\pi$

Because $\tan \theta$ is also given by $\frac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x} = \frac{k_y}{k_x} \Rightarrow \Delta y = \frac{k_y}{k_x} \Delta x$$

Substituting for $\Delta y$ yields:

$$k_x \Delta x + k_y \frac{k_y}{k_x} \Delta x = 2\pi$$

Solve for $\Delta x$ to obtain:

$$\Delta x = \frac{2\pi k_x}{k_x^2 + k_y^2}$$

Similarly:

$$\Delta y = \frac{2\pi k_y}{k_x^2 + k_y^2}$$

Substitute for $\Delta x$ and $\Delta y$ in equation (1) to obtain:

$$\lambda = \sqrt{\left(\frac{2\pi k_x}{k_x^2 + k_y^2}\right)^2 + \left(\frac{2\pi k_y}{k_x^2 + k_y^2}\right)^2} = \frac{2\pi}{\sqrt{k_x^2 + k_y^2}}$$

Substituting for $\lambda$ in equation (2) yields:

$$\nu = \frac{\omega}{2\pi} \frac{2\pi}{\sqrt{k_x^2 + k_y^2}} = \frac{\omega}{\sqrt{k_x^2 + k_y^2}}$$

81 •• [SSM] The speed of sound in air is proportional to the square root of the absolute temperature $T$ (Equation 15-5). (a) Show that if the air temperature changes by a small amount, the fractional change in the fundamental frequency of an organ pipe is approximately equal to half the fractional change in the absolute temperature. That is, show that $\frac{\Delta f}{f} \approx \frac{1}{2} \frac{\Delta T}{T}$, where $f$ is the frequency at absolute temperature $T$ and $\Delta f$ is the change in frequency when the temperature changes by $\Delta T$. (Ignore any change in the length of the pipe due to thermal expansion of the organ pipe.) (b) Suppose that an organ pipe that is stopped at one end has a fundamental frequency of 200.0 Hz when the temperature is 20.00ºC. Use the approximate result from Part (a) to determine its fundamental frequency when the temperature is 30.00ºC. (c) Compare your Part (b) result to what you would get
using exact calculations. (Ignore any change in the length of the pipe due to thermal expansion.)

**Picture the Problem** We can express the fundamental frequency of the organ pipe as a function of the air temperature and differentiate this expression with respect to the temperature to express the rate at which the frequency changes with respect to temperature. For changes in temperature that are small compared to the temperature, we can approximate the differential changes in frequency and temperature with finite changes to complete the derivation of $\Delta f/f = \frac{1}{2} \Delta T/T$. In Part (b) we’ll use this relationship and the data for the frequency at 20.00°C to find the frequency of the fundamental at 30.00°C.

(a) Express the fundamental frequency of an organ pipe in terms of its wavelength and the speed of sound:

$$f = \frac{v}{\lambda}$$

Relate the speed of sound in air to the absolute temperature:

$$v = \sqrt{\frac{RT}{M}} = C \sqrt{T}$$

where

$$C = \sqrt{\frac{\gamma R}{M}} = \text{constant}$$

Defining a new constant $C'$, substitute to obtain:

$$f = \frac{C}{\lambda} \sqrt{T} = C' \sqrt{T}$$

because $\lambda$ is constant for the fundamental frequency we ignore any change in the length of the pipe.

Differentiate this expression with respect to $T$:

$$\frac{df}{dT} = \frac{1}{2} C' T^{-1/2} = \frac{f}{2T}$$

Separate the variables to obtain:

$$\frac{df}{f} = \frac{dT}{2T}$$

For $\Delta T \ll T$, we can approximate $df$ by $\Delta f$ and $dT$ by $\Delta T$ to obtain:

$$\frac{\Delta f}{f} = \frac{1}{2} \frac{\Delta T}{T}$$

(b) Express the fundamental frequency at 30.00°C in terms of its frequency at 20.00°C:

$$f_{30} = f_{20} + \Delta f$$
Chapter 16

Solve the result in (a) for $\Delta f$:

$$\Delta f = \frac{1}{2} f \frac{\Delta T}{T}$$

Substitute for $\Delta f$ to obtain:

$$f_{30} = f_{20} + \frac{1}{2} f_{20} \frac{\Delta T}{T}$$

Substitute numerical values and evaluate $f_{30}$:

$$f_{30} = 200.0 \text{ Hz}$$

$$+ \frac{1}{2} (200.0 \text{ Hz}) \frac{(303.15 \text{ K} - 293.15 \text{ K})}{293.15 \text{ K}}$$

$$= 203.4 \text{ Hz}$$

(c) The exact expression for $f_{30}$ is:

$$f_{30} = \frac{v_{30}}{\lambda} = \sqrt{\frac{\gamma R T_{30}}{M}} \frac{\lambda}{\lambda} = \sqrt{\frac{\gamma R T_{30}}{\lambda^2 M}}$$

The exact expression for $f_{20}$ is:

$$f_{20} = \frac{v_{20}}{\lambda} = \sqrt{\frac{\gamma R T_{20}}{M}} \frac{\lambda}{\lambda} = \sqrt{\frac{\gamma R T_{20}}{\lambda^2 M}}$$

Dividing the first of these equations by the second and simplifying yields:

$$\frac{f_{30}}{f_{20}} = \sqrt{\frac{T_{30}}{T_{20}}}$$

Solve for $f_{30}$ to obtain:

$$f_{30} = \frac{T_{30}}{T_{20}} f_{20}$$

Substitute numerical values and evaluate $f_{30}$:

$$f_{30} = \sqrt{\frac{303.15 \text{ K}}{293.15 \text{ K}}} (200.0 \text{ Hz})$$

$$= 203.4 \text{ Hz}$$

The pipe in Figure 16-36 is kept filled with natural gas (methane, CH$_4$). The pipe is punctured with a line of small holes 1.00 cm apart down its entire 2.20 m length. A speaker forms the closure on one end of the pipe, and a solid piece of metal closes the other end. What frequency is being played in this picture? The speed of sound in low pressure methane at room temperature is about 460 m/s.

**Picture the Problem** The frequency of the sound being played is given by $f = \frac{v}{\lambda}$ . Because the holes are not visible in the photograph, we can not use their
separation to determine the wavelength of the sound being played. Instead, we can use the fact that the separation of the leftmost and rightmost flame maxima is 6 quarter wavelengths (drawing a standing wave pattern that is consistent with the flame pattern shown in Figure 16-36 will help you see this). Using any convenient scale to measure \( s_1 \) and \( s_2 \) (scaled distances on the photograph) and setting up a proportion involving these distances, the actual length \( L \) of the pipe, and the number of wavelengths in the distance \( s_1 \) will yield \( \lambda \) and, hence, \( f \).

\[
\begin{align*}
s_1 & \quad | \quad L \quad | \quad s_2
\end{align*}
\]

The frequency of the sound is given by:

\[
f = \frac{v}{\lambda}
\]

Letting \( L \) represent the length of the pipe yields:

\[
\frac{\frac{\lambda}{s_1}}{\frac{\lambda}{s_2}} = \frac{L}{s_2} \Rightarrow \lambda = \frac{2Ls_1}{3s_2}
\]

Substituting for \( \lambda \) yields:

\[
f = \frac{v}{2Ls_1} = \frac{3vs_2}{2Ls_1}
\]

If \( s_1 = 10.5 \text{ cm} \) and \( s_2 = 14.5 \text{ cm} \), then:

\[
f = \frac{3(460 \text{ m/s})(14.5 \text{ cm})}{2(2.20 \text{ m})(10.5 \text{ cm})} = 433 \text{ Hz}
\]

**83** Assume that your clarinet is entirely filled with helium and that before you start to play you fill your lungs with helium. You pick up the clarinet and play it as though you were trying to play a B-flat, which has a frequency of 277 Hz. The frequency of 277 Hz is the natural resonance frequency of this clarinet with all finger holes closed and when filled with air. What frequency do you actually hear?

**Picture the Problem** The resonance frequency of the clarinet depends on the nature of the gas with which it is filled. We can express this frequency for both air and helium and express their ratio to eliminate the constant factors \( \lambda \), \( R \), and \( T \). See Appendix C for the molar masses of helium and air.

Express the frequency of B-flat in helium:

\[
f_{\text{B-flat,He}} = \frac{v_{\text{He}}}{\lambda} = \frac{1}{\lambda} \sqrt{\frac{\gamma_{\text{He}}RT}{M_{\text{He}}}}
\]
Divide the second these equations by the first and simplify to obtain:

\[
\frac{f_{\text{B-flat, He}}}{f_{\text{B-flat, air}}} = \frac{1}{\lambda} \sqrt{\frac{\gamma_{\text{He}} RT}{M_{\text{He}}}} = \frac{1}{\lambda} \sqrt{\frac{\gamma_{\text{air}} RT}{M_{\text{air}}}} = \frac{M_{\text{air}} \gamma_{\text{He}}}{M_{\text{He}} \gamma_{\text{air}}}
\]

Solving for \(f_{\text{B-flat, He}}\) yields:

\[
f_{\text{B-flat, He}} = \sqrt{\frac{M_{\text{air}} \gamma_{\text{He}}}{M_{\text{He}} \gamma_{\text{air}}}} f_{\text{B-flat, air}}
\]

Substitute numerical values and evaluate \(f_{\text{B-flat, He}}\):

\[
f_{\text{B-flat, He}} = \sqrt{\frac{(28.81 \text{ g/mol})(1.67)}{(4.003 \text{ g/mol})(1.4)}} (277 \text{ Hz}) = 812 \text{ Hz}
\]

84  ... A 2.00-m-long wire that is fixed at both ends is vibrating in its fundamental mode. The tension in the wire is 40.0 N and the mass of the wire is 0.100 kg. At the midpoint of the wire, the amplitude is 2.00 cm. (a) Find the maximum kinetic energy of the wire. (b) What is the kinetic energy of the wire at the instant the transverse displacement is given by \(y = 0.0200 \sin \left( \frac{\pi}{L} x \right) \), where \(y\) is in meters if \(x\) is in meters, for 0.00 m \(\leq x \leq 2.00 \text{ m}\)? (c) For what value of \(x\) is the average value of the kinetic energy per unit length the greatest? (d) For what value of \(x\) does the elastic potential energy per unit length have its maximum value?

**Picture the Problem** The maximum kinetic energy of the wire is given by \(K_{\text{max}} = \frac{1}{4} m \omega^2 A^2\). We can use \(v = f \lambda\) and \(v = \sqrt{F_0 / \mu}\) to find an expression for \(\omega\).

In Part (d) we’ll use \(\frac{\Delta U}{\Delta x} \approx \frac{1}{4} F_0 \left( \frac{\partial y}{\partial x} \right)^2\) (Problem 15-104) to determine where the potential energy per unit length has its maximum value.

(a) The maximum kinetic energy of the wire is given by:

\[
K_{\text{max}} = \frac{1}{4} m \omega^2 A^2
\]

(b) Express \(\omega_1\) in terms of \(f_1\):

\[
\omega_1 = 2 \pi f_1
\]

(c) Relate \(f_1\) to the speed of transverse waves on the wire and the wavelength of the fundamental mode:

\[
f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}
\]

where \(L\) is the length of the wire.
Express the speed of the transverse waves on the wire in terms of the tension in the wire:

\[ v = \sqrt{\frac{F_t}{\mu}} = \sqrt{\frac{F_t L}{m}} \]

Substitute and simplify to obtain:

\[ f_1 = \frac{1}{2L} \sqrt{\frac{F_t L}{m}} = \sqrt{\frac{F_t}{4mL}} \]

Substitute for \( \omega_1 \) and \( f_1 \) in equation (1) to obtain:

\[ K_{\text{max}} = \frac{1}{4}m \left[ 2\pi \sqrt{\frac{F_t}{4mL}} \right]^2 A^2 = \frac{\pi^2 F_t}{4L} A^2 \]

Substitute numerical values and evaluate \( K_{\text{max}} \):

\[ K_{\text{max}} = \frac{\pi^2 (40.0 \text{ N})}{4(2.00 \text{ m})} (2.00 \times 10^{-2} \text{ m})^2 \]

\[ = 19.7 \text{ mJ} \]

(b) Express the wave function for a standing wave in its first harmonic:

\[ y_1(x,t) = A_i \sin k_i x \cos \omega_1 t \quad (2) \]

At the instant the transverse displacement is given by

\( (0.02 \text{ m}) \sin (\pi x/2) \):

\[ \cos \omega_1 t = 1 \Rightarrow \omega_1 t = 0 \]

and

\[ K = 0 \]

(c) \( dK \) is a maximum where the displacement of the wire is greatest; that is, at its midpoint:

\[ x = \frac{1}{2} L = \frac{1}{2} (2.00 \text{ m}) = 1.00 \text{ m} \]

(d) From Problem 15-104:

\[ \frac{\Delta U}{\Delta x} \approx \frac{1}{2} F_t \left( \frac{\partial y}{\partial x} \right)^2 \]

Express the condition on \( \frac{\partial y}{\partial x} \)

\[ \frac{\partial y}{\partial x} = \left( \frac{\partial y}{\partial x} \right)_{\text{max}} \]

Differentiate

\[ y_i(x,t) = A_i \sin k_i x \cos \omega_1 t \]

with respect to \( x \) and set the derivative equal to zero for extrema:

\[ \frac{\partial y_i}{\partial x} = \frac{\partial}{\partial x} (A_i \sin k_i x \cos \omega_1 t) \]

\[ = k_i A_i \cos k_i x \cos \omega_1 t \]

\[ = 0 \]

or

\[ \cos k_i x = 0 \]
Solve for $k_1x$ and then $x$:
\[ k_1x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2k_1} = \frac{\pi\lambda}{2(2\pi)} = \frac{\lambda}{4} \]

Because $\lambda = 2L$ :
\[ x = \frac{1}{2}L \Rightarrow x = \frac{1}{2}(2.00\, \text{m}) = 1.00\, \text{m} \]
That is, the potential energy per unit length is a maximum at the midpoint of the wire.

Remarks: In Part (d) we’ve shown that $\Delta U/\Delta x$ has an extreme value at $x = 1$ m. To show that $\Delta U/\Delta x$ is a maximum at this location, you need to examine the sign of the 2nd derivative of $y_1(x, t)$ at this point.

85  SSM  In principle, a wave with almost any arbitrary shape can be expressed as a sum of harmonic waves of different frequencies. (a) Consider the function defined by
\[
 f(x) = \frac{4}{\pi} \left( \cos x - \cos \frac{3x}{3} + \cos \frac{5x}{5} - \cdots \right) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos[(2n+1)x]}{2n+1}
\]
Write a spreadsheet program to calculate this series using a finite number of terms, and make three graphs of the function in the range $x = 0$ to $x = 4\pi$. To create the first graph, for each value of $x$ that you plot, approximate the sum from $n = 0$ to $n = \infty$ with the first term of the sum. To create the second and third graphs, use only the first five term and the first ten terms, respectively. This function is sometimes called the square wave. (b) What is the relation between this function and Liebnitz’ series for $\pi$, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$?

A spreadsheet program to evaluate $f(x)$ is shown below. Typical cell formulas used are shown in the table.

<table>
<thead>
<tr>
<th>Cell</th>
<th>Content/Formula</th>
<th>Algebraic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>A6</td>
<td>A5+0.1</td>
<td>$x + \Delta x$</td>
</tr>
<tr>
<td>B4</td>
<td>2*B3+1</td>
<td>$2n+1$</td>
</tr>
<tr>
<td>B5</td>
<td>(-1)^B$3$<em>COS(B$4</em>$A5)/B$4$*4/PI()</td>
<td>$4 \left( \cos((1)(0.0)) \right) \over \pi$</td>
</tr>
<tr>
<td>C5</td>
<td>B5+(-1)^C$3$<em>COS(C$4</em>$A5)/C$4$*4/PI()</td>
<td>$1.2732 + \left( \cos((3)(0.0)) \right) \over \pi$</td>
</tr>
</tbody>
</table>
The graph of $f(x)$ versus $x$ for $n = 1$ follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>K</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$n = 0$</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>$2n+1=$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>19</td>
</tr>
<tr>
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<td>0.8544</td>
<td>1.0952</td>
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<td>1.0422</td>
<td>0.8990</td>
<td>1.0261</td>
</tr>
</tbody>
</table>
The graph of $f(x)$ versus $x$ for $n = 5$ follows:

![Graph for n=5]

The graph of $f(x)$ versus $x$ for $n = 10$ follows:

![Graph for n=10]

Evaluate $f(2\pi)$ to obtain:

\[
\begin{align*}
f(2\pi) &= \frac{4}{\pi}\left(\frac{\cos 2\pi}{1} - \frac{\cos 3(2\pi)}{3} + \frac{\cos 5(2\pi)}{5} - \ldots\right) \\
&= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \\
&= 1
\end{align*}
\]

which is equivalent to the Liebnitz formula.
86 Write a **spreadsheet program** to calculate and graph the function

\[
y(x) = \frac{4}{\pi} \left( \sin x - \frac{\sin 3x}{9} + \frac{\sin 5x}{25} - \cdots \right) = \frac{4}{\pi} \sum_{n} \frac{(-1)^n \sin(2n+1)x}{(2n+1)^2}
\]
for \(0 \leq x \leq 4\pi\). Use only the first 25 terms in the sum for each value of \(x\) that you plot.

A spreadsheet program to evaluate \(y(x)\) is shown below. Typical cell formulas used are shown in the table.

<table>
<thead>
<tr>
<th>Cell</th>
<th>Content/Formula</th>
<th>Algebraic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>A6</td>
<td>A5+0.1</td>
<td>(x + \Delta x)</td>
</tr>
<tr>
<td>B4</td>
<td>2*B3+1</td>
<td>(2n + 1)</td>
</tr>
<tr>
<td>B5</td>
<td>((-1)^n<em>B4</em>\sin(B4<em>A5)/(B4)^2</em>4/\pi()))</td>
<td>(\frac{4 \cdot (-1)^{n} \sin(0.0)}{\pi \cdot (1)^2})</td>
</tr>
<tr>
<td>B6</td>
<td>((-1)^n<em>B4</em>\sin(B4<em>A6)/(B4)^2</em>4/\pi()))</td>
<td>(\frac{4 \cdot (-1)^{n} \sin(0.1)}{\pi \cdot (1)^2})</td>
</tr>
<tr>
<td>C5</td>
<td>B5+((-1)^n<em>C4</em>\sin(C4<em>A5)/(C4)^2</em>4/\pi()))</td>
<td>(0 + \frac{4 \cdot (-1)^{n} \sin(0.1)}{\pi \cdot (1)^2})</td>
</tr>
<tr>
<td>C6</td>
<td>B6+((-1)^n<em>C4</em>\sin(C4<em>A6)/(C4)^2</em>4/\pi()))</td>
<td>(0.1271 + \frac{4 \cdot (-1)^{n} \sin(3)(0.1)}{\pi \cdot (3)^2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>K</th>
<th>L</th>
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<tbody>
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<td>1</td>
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<tr>
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<td>(n=)</td>
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<td>1</td>
<td>2</td>
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<td>10</td>
</tr>
<tr>
<td>4</td>
<td>(2n+1=)</td>
<td>1</td>
<td>3</td>
<td>5</td>
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<td>0.6005</td>
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<td>0.8380</td>
<td>0.7968</td>
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</tbody>
</table>
The following graph was obtained using $n = 10$.

![Graph](image)

If you clap your hands at the end of a long, cylindrical tube, the echo you hear back will not sound like the handclap; instead, you will hear what sounds like a whistle, initially at a very high frequency, but descending rapidly down to almost nothing. This "culvert whistler" is easily explained if you think of the sound from the clap as a single compression radiating outward from the hands. The echoes of the handclap arriving at your ear have traveled along different paths through the tube, as shown in Figure 16-37. The first echo to arrive travels straight down and straight back along the tube, while the second echo reflects once off of the center of the tube going out, and again going back, the third echo reflects twice at points 1/4 and 3/4 of the distance, etc. The tone of the sound you hear reflects the frequency at which these echoes reach your ears. 

(a) Show that the time delay between the $n_{\text{th}}$ echo and the $(n+1)_{\text{th}}$ is

$$\Delta t_n = \frac{2}{v} \left( \sqrt{(2n)^2 r^2 + L^2} - \sqrt{(2(n-1))^2 r^2 + L^2} \right)$$

where $v$ is the speed of sound, $L$ is the length of the tube and $r$ is its radius.

(b) Using a spreadsheet program or graphing calculator, graph $\Delta t_n$ versus $n$ for $L = 90.0$ m, $r = 1.00$ m. (These values are the approximate length and radius of the long tube in the San Francisco Exploratorium.) Go to at least $n = 100$.

(c) From your graph, explain why the frequency decreases over time. What are the highest and lowest frequencies you will hear in the whistler?

**Picture the Problem** From the diagram above, the $n_{\text{th}}$ echo will reflect $n - 1$ times going out, and the same number of times going back. If we "unfold" the ray into a straight line, we get the representation shown below. Using this figure we can express the distance $d_n$ traveled by the $n_{\text{th}}$ echo and then use this result to express the time delay between the $n_{\text{th}}$ and $(n+1)_{\text{th}}$ echoes. The reciprocal of this time delay is the frequency corresponding to the $n_{\text{th}}$ echo.
Apply the Pythagorean theorem to the right triangle whose base is \( L \), whose height is \( 2(n - 1) \), and whose hypotenuse is \( d_n \) to obtain:

\[
d_n = 2\sqrt{4(n-1)^2r^2 + L^2}
\]

Express the time delay between the \( n_{th} \) and \( (n+1)_{th} \) echoes:

\[
\Delta t_n = \frac{d_n}{v}
\]

Substitute to obtain:

\[
\Delta t_n = \frac{2}{v} \left( \sqrt{(2n)^2 r^2 + L^2} - \sqrt{(2n-1)^2 r^2 + L^2} \right)
\]

(b) A spreadsheet program to calculate \( \Delta t_n \) as a function of \( n \) is shown below. The constants and cell formulas used are shown in the table.
The graph of $\Delta t_n$ as a function of $n$ shown below was plotted using the data from columns B and D.

(c) The frequency heard at any time is $1/\Delta t_n$, so because $\Delta t_n$ increases over time, the frequency of the culvert whistler decreases.

The highest frequency corresponds to $n = 1$ and is given by:

$$f_{\text{highest}} = \frac{1}{\Delta t_1}$$

Substitute for $\Delta t_1$ to obtain:

$$f_{\text{highest}} = \frac{1}{\Delta t_1} = \frac{v}{2\sqrt{(2)^2 r^2 + L^2 - \sqrt{L^2}}}$$

Substitute numerical values and evaluate $f_{\text{highest}}$:

$$f_{\text{highest}} = \frac{343 \text{ m/s}}{2\left(\sqrt{4(1.00 \text{ m})^2 + (90.0 \text{ m})^2 - 90.0 \text{ m}}\right)} = 7.72 \text{ kHz}$$
The lowest frequency end can be found by examining the limit of $\Delta t_n$ as $n \to \infty$:

$$
\lim_{n \to \infty} \Delta t_n = \lim_{n \to \infty} \left[ \frac{2}{v} \left( \frac{L}{2n} \right) \sqrt{r^2 + \frac{L^2}{(2n)^2}} - 2(n-1) \sqrt{r^2 + \frac{L^2}{(2(n-1))^2}} \right] \\
= \frac{2r}{v} (2n - 2n + 2) = \frac{4r}{v} 
$$

Express $f_{\text{lowest}}$ in terms of $\Delta t_\infty$:

$$
f_{\text{lowest}} = \frac{1}{\Delta t_\infty} = \frac{v}{4r} 
$$

Substitute numerical values and evaluate $f_{\text{lowest}}$:

$$
f_{\text{lowest}} = \frac{343 \text{ m/s}}{4(1.00 \text{ m})} = 85.8 \text{ Hz} 
$$